GENTZEN SYSTEMS WITH NEGATION
AND
CRAIG FACTORIZATION IN THE ASSOCIATED CATEGORIES

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ABSTRACT

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Gentzen systems with negation, and Craig factorization in the associated categories

The syntactic and semantic properties of the Gentzen systems LM, LD, LJ, LE and LK are studied, and the systems are shown to be correct, complete and compact relative to the model theory developed. Craig's interpolation theorem is proved for LM, LJ, LE and LK, and for the propositional fragment of LD. The propositional fragments of the systems are interpreted as structured categories in the manner of Szabo and Lambek, and Craig's theorem is shown to be stable under this interpretation, thus providing a factorization procedure for morphisms.

CONTENTS

Introduction

This thesis is intended to initiate an enquiry into the meaning of negation in extensions of the positive fragment of Gentzen's intuitionist claculus LJ, and is motivated by the investigations of H.B. Curry in Curry (52), (52b), (52c), (57) and (63). By negation we understand a unary operation on the set of formulae, the negation of a formula A being denoted by $\neg A$, with the intended interpretation that $\neg A$ is a theorem if and only if A is false. It would seem that the weakest conditions which can be imposed on such an operation while retaining some connexion with our intuitive notions of truth and falsity are the following.

(a) The conjunction of any formula and its negation shall be false, in symbols:

$$\longrightarrow \neg(A \wedge \neg A).$$

(b) Any formula from which a false formula can be derived shall itself be false, in symbols:

$$\frac{A \longrightarrow B \qquad \longrightarrow \neg B}{\longrightarrow \neg A}$$

Throughout our work negation will be defined by introducing into the language a fixed atomic formula $\perp$ to stand for "the false", and by taking A as an abbreviation for $A \supset \perp$. Clearly, the negation operator so defined satisfies both of the requirements above. The study of negation may thus be construed as the study of "the false". Taking $\perp$ as a fixed atomic formula with no additional requirements yields the

the system of "minimal logic" LM (see Prawitz (65), and Prawitz and Malmnäs (68)). Thus we may reformulate the object of our enquiries as the study of extensions of the deductive system LM. The systems LD, LJ, LE and LK are such extensions. The requirement that $\perp$ be an atomic formula was made in order to avoid the trivial case in which $\perp$ is itself a theorem. In this case the negation of any formula would be a theorem, and so negation becomes uninteresting. In Section 1 we define the five systems for negation LM, LD, LJ, LE and LK (the systems LJ and LK coincide, with minor modifications, with their Gentzen namesakes), and in Section 2 we go on to develop a model theory for these systems adapted from Rasiowa and Sikorski (63). The particular interest of the systems LM, LD, LJ and LK derives from the fact that each is, in a certain sense, characterized by a well-known principle of negation. Thus, LM is characterized by the "principle of non-contradiction", LD is characterized by the "law of the excluded middle", LJ is characterized by intuitionist negation, and LK is characterized by classical negation. LE is a system, intermediate between LD and LK, which might be called a system of "classical minimal logic". These systems do not, of course, exhaust the interesting possibilities. For example, recent results of Szabo (a) suggest the interest of examining the systems characterized by the various de Morgan laws for negation.

Lambek (68), (69), (72) and (a), and Szabo (74), (c),

(d) and (e) have developed methods for representing the propositional and first order Gentzen systems LJ and LK as Cartesian closed categories, which may be regarded as generalizations of the associated Lindenbaum algebras. The question arises as to whether this can also be done in an interesting way for the systems LM, LD and LE. We provide such an interpretation for the propositional fragments of these systems in Section 4. The ease with which the representation is constructed suggests that category theory provides a natural framework for developing the model theory of arbitrary systems for negation (i.e. arbitrary extensions of LM "lying below" LK).

Because of the richer structure of the representing categories more detail of the structure of the systems "lifts" to the categories than is the case for the Lindenbaum algebras. In particular, while the transition to the Lindenbaum algebras results in the loss of all information concerning the derivation of a given derivable sequent, in the categories equivalence classes of derivations are retained as morphisms. That this is so raises the question of how much of the proof theory of the systems "lifts" to the representing categories. In this thesis we answer this question for Craig's Interpolation Theorem (Craig (57)), one of the standard results in proof theory. For Gentzen systems this theorem states that if a sequent $A \longrightarrow B$ is derivable then there is an "interpolation formula" J constructed from the variables and relation symbols occurring in A and B, so that the sequents $A \longrightarrow J$ and $J \longrightarrow B$ are derivable. The theorem "lifts" in a trivial sense to the Lindenbaum algebras. However, interpreted categorically, the theorem requires the commutativity of certain diagrams and is, therefore, more interesting; the proofs are given in Section 4.

Section 3 is devoted to a proof of Craig's theorem for each of the systems LM, LJ, LE and LK. For LD, unfortunately, we have only been able to establish the theorem for the propositional fragment, although this is not serious since in any case we only provide a categorical representation for the propositional fragments of the systems.

Intuitively, we have come to think of the categories as simplifications of the deductive systems, and of the Lindenbaum algebras as simplifications of the categories, and we visualize a chain

$$\text{Systems} \longrightarrow \text{Categories} \longrightarrow \text{Lindenbaum algebras}$$

in which all suitably interpreted theorems of proof theory "lift".

As far as notation is concerned we have followed standard practice where this is established, otherwise we have been eclectic, aiming for consistency and clarity. In the latter regard, it is, perhaps, fair to warn the reader that we employ the symbol 0 to denote both the number zero and the empty set, and that if S is a subset of the domain of a function f, the notation $f \upharpoonright S$ will denote "the restriction of f to S". In addition we make free use of the abbreviation "iff" for "if and only if". Bibliographical conventions are explained at the beginning of the bibliography.

It is a great pleasure for me to be able to acknowledge publicly the enormous debt I owe Dr. Manfred E. Szabo for his continued support and encouragement in this and other undertakings.

Section 1. The Systems LM, LD, LJ, LE and LK.

1.1 Introduction

In this section we present the definitions of the (first-order Gentzen) language L, and the five deductive systems LM, LD, LJ, LE and LK. For our purposes we think of a <u>first-order Gentzen language</u> as an ordered quadruple

$$\langle A, T, F, S\rangle$$

of sets A, T, F and S, the elements of which are called symbols, terms, formulae and sequents, respectively. The definition of our particular language L

$$L = \langle A_L, T_L, F_L, S_L\rangle$$

follows immediately.

1.2 Definition

Let V be a non-empty, transitive set which is closed under unions and pair formation; i.e. V satisfies

(i) $V \neq 0$,

(ii) if $x \in y$ and $y \in V$ then $x \in V$,

(iii) if $x \in V$ then $\bigcup x \in V$,

and (iv) if $x, y \in V$ then $\langle x,y\rangle = \{\{x\},\{x,y\}\} \in V$.

Such sets exist in any desired cardinality, indeed it suffices to take for V the set of all finite subsets of $\mathfrak{m}$, where $\mathfrak{m}$ is some infinite cardinal. V is called the <u>underlying set of the language L</u>.

By the axiom of regularity there is some $x \in V$ so that $x \cap V = 0$, and this implies, since V is transitive, that $x = 0$.

Because of the equality

$$x \cup \{x\} = \bigcup \bigcup \{\{x\}, \{x, \bigcup <x,x>\}\},$$

it is clear that if $x \in V$, then the successor of x is also an element of V. We have now shown that $\omega \subseteq V$.

1.3 Definitions

We specify the following subsets of V:

(i) $FV = \{<0,v> : v \in V\}$;

(ii) $BV = \{<1,v> : v \in V\}$;

(iii) $C = \{<2,v> : v \in V\}$;

(iv) for each $n \in \omega$, $R^n = \{<4,n,v> : v \in V\}$;

and (v) $F = \{<5,0>\}$.

The elements of these sets will be refered to as "free variables", "bound variables", "constants", "n-ary relation symbols" and "the false", respectively. Informally, free variables will be denoted by the letters x, y, ...; bound variables by ξ, η, ...; constants by a, b, ...; n-ary relation symbols by A^n, B^n, ...; and, the false by $\perp$.

We now define the set of symbols of L, A_L, by

$$A_L = FV \cup BV \cup C \cup (\bigcup_{n \in \omega} R^n) \cup F,$$

the set of terms of L, T_L, by

$$T_L = FV \cup C,$$

and the set of atomic formulae of L, At_L, by

$$At_L = \bigcup_{n \in \omega} (R^n \times (T_L)^n) \cup F.$$

1.4 Definition

The set F_L of formulae of L is defined inductively by

(i) $At_L \subseteq F_L$;

(ii) if $A, B \in F_L$ then $\langle 8,A,B \rangle$, $\langle 9,A,B \rangle$, $\langle 10,A,B \rangle \in F_L$;

(iii) if $A \in F_L$, $t \in T_L$ and $\xi \in BV$, then $\langle 11, \xi, A_t(\xi) \rangle$, $\langle 12, \xi, A_t(\xi) \rangle \in F_L$,

where $A_t(\xi)$ denotes the result of replacing t wherever it occurs, if at all, in A by ξ.

From this point on capital Roman letters A, B, ... will denote arbitrary formulae. The notation $A_t(\xi)$ above is extended in an obvious way to $A_s(s')$ where s and s' are arbitrary terms or bound variables. Informally $\langle 8,A,B \rangle$, $\langle 9,A,B \rangle$, $\langle 10,A,B \rangle$, $\langle 11, \xi, A_t(\xi) \rangle$ and $\langle 12, \xi, A_t(\xi) \rangle$ will be denoted by $A \wedge B$, $A \vee B$, $A \supset B$, $\forall \xi\, A_t(\xi)$ and $\exists \xi\, A_t(\xi)$ respectively, parentheses being added where necessary to preserve unique readability. We also abbreviate $\langle 10, A, \perp \rangle$ by $\neg A$.

1.5 Definitions

The set S_L of <u>sequents of L</u> is defined by

$$S_L = \bigcup_{n \geq 1} F_L{}^n.$$

If $\langle A_1, \ldots, A_{n-1}, A_n \rangle$ is a sequent then A_n is referred to as the <u>succedent</u>, and $\langle A_1, \ldots, A_{n-1} \rangle$ as the <u>antecedent</u>. Note that a sequent always has a succedent but may fail to have an antecedent, in the latter event we shall say that "the antecedent is empty". Sequents will usually be written $\Gamma \longrightarrow A$, where Γ is the antecedent and A is the succedent. In future capital Greek letters, unless otherwise specified or made clear by the context, will denote finite, possibly empty, sequences of formulae, and we use the notation $A \in \Gamma$ to indicate that the formula A appears in Γ. Occasionally we shall even write $x \in \Gamma$ to indicate that the symbol $x \in A_B$ appears in a formula appearing in Γ, and $\Delta \subseteq \Gamma$ to indicate that every formula appearing in Δ appears also in Γ.

We also define a sentence to be a formula in which no element of FV appears.

1.6 Definition

By a Gentzen type first order deductive system LX we understand an ordered quadruple

$$\langle L, Ax(X), Rule(X), Der(X) \rangle,$$

where L is a language of the form described in paragraphs 1.1 to 1.5 above. In everything that follows we shall assume that some such language has been chosen and fixed. $Ax(X) \subseteq S_L$ is called the set of axioms for LX. Rule(X) is a set of relations with domain S_L called rules of inference for LX; for simplicity's sake we shall restrict our attention to binary and ternary rules. Der(X) is a set of ordered pairs, to be defined in detail below, called derivations of LX.

Elements of $r \in Rule(X)$ are called applications of r. If $\langle S, S' \rangle \in r \in Rule(X)$, we also say that "$\langle S, S' \rangle$ is an application of the rule r to the premise S with conclusion S'". In a similar way $\langle S, S'', S' \rangle \in r'$ is said to be "an application of the rule r' to the premises S and S" with conclusion S'". When we have occasion to write down specific applications of rules, we shall normally do so informally by writing the premises (of the application) above a horizontal line, with the name of the rule written to the right of it and the conclusion below, viz.

$$\frac{S}{S''}\, r, \text{ or } \frac{S \qquad S''}{S'}\, r'.$$

1.7 Definition

In order to define Der(X) we introduce the notion of a (dyadic ordered) tree.

A <u>tree</u> T is an ordered quadruple

$$\langle S, s, e, \theta \rangle,$$

satisfying:

(i) S is a finite non-empty set of elements called <u>points</u>, with a distinguished element e called the <u>endpoint</u>.

(ii) $s: S-\{e\} \longrightarrow S$ is a function so that for each point x, $s^{-1}(x)$ contains at most two points. For each point x, $s(x)$ is called the <u>successor</u> of x, and points in $s^{-1}(x)$ are called <u>predecessors</u> of x. A point with no predecessors is an <u>initial point</u>, with one predecessor a <u>simple point</u>, and with two predecessors a <u>junction point</u>.

(iii) $\theta: J \longrightarrow S \times S$, where J is the set of junction points, is a function which assigns to each junction point x an ordered pair $\langle y,z \rangle$, where y and z are the predecessors of x. If $\theta(x) = \langle y,z \rangle$, y is called the <u>left predecessor of x</u>, and z the <u>right predecessor of x</u>.

A <u>subtree</u> T' of T is defined to be a tree $\langle S',s',e',\theta' \rangle$ with junction points J', for which $S' \subseteq S$, $e' = e$, $s' = s \restriction S'$, and $\theta' = \theta \restriction J'$.

1.8 Definition

A <u>derivation</u> $f \in Der(X)$ is defined to be an ordered pair $\langle T, p \rangle$, where T is a tree $\langle S,s,e,\theta \rangle$, and p is a function $p:S \longrightarrow S_L$ satisfying:

(i) If x is an initial point of T then $p(x) \in Ax(X)$. p(x) is called an <u>initial sequent</u> of f.

(ii) If x is a simple point and $x = s(y)$ then $\langle p(y),p(x) \rangle \in r$, for some binary rule r.

(iii) If x is a junction point and $\theta(x) = \langle y,z \rangle$ then $\langle p(y),p(z),p(x) \rangle \in r'$, for some ternary $r' \in Rule(X)$.

p(e) is called the <u>end-sequent</u> of f. Derivations will usually be denoted by the letters f, g, h with subscripts or primes where necessary (the reason for this choice of notation will become apparent in Section 4). Whenever we have occasion to write out a derivation of LX, we shall usually do so informally by writing down each application of a rule occurring in the derivation informally (see definition 1.6) in such a way that for each point x, p(x) is written precisely once (for an example of how this is done see theorem 1.19 below).

A sequent S is said to be <u>derivable</u> in the system LX provided there is a derivation $f \in Der(X)$ for which S is the end-sequent, in this case we also say that "f is a derivation of S". By abuse of language we shall also write "$S \in Der(X)$" as an abbreviation for "S is a derivable sequent in LX".

1.9 Remark

We notice that by definition 1.8 Der(X) is completely determined by the sets Ax(X) and Rule(X) for any system LX. Since we are holding the language L fixed, it follows that any system LX may be completely specified by describing the sets Ax(X) and Rule(X).

1.10 Definitions

A binary relation r on S_L is said to be a <u>derived rule</u> of an arbitrary system LX, if whenever $\langle S,S' \rangle \in r$, and there is a derivation $f \in Der(X)$ of the sequent S, then there is also some $g \in Der(X)$ which is a derivation of S'. Derived ternary rules are defined similarly.

A system LY is said to be an <u>extension</u> of a system LX provided every sequent derivable in LX is also derivable in LY. If LY is an extension of LX and LX is an extension of LY then we say that the systems are (proof theoretically) <u>equivalent</u>. All the systems with which we shall have to deal will be extensions of the system LM which we define next.

1.11 Definition

The system LM is defined by

(1) $Ax(M) = \{\langle A_1, \ldots, A_n\rangle \in S_L : A_n = A_{n-1}\}$,

(2) $Rule(M) = \{\to\wedge, \wedge\to, \to\vee, \vee\to, \to\supset, \supset\to, \to\forall,$
$\forall\to, \to\exists, \exists\to, \theta, \kappa, \pi\}$.

The elements of Rule(M) are defined by the following, where Γ, Δ and Θ are arbitrary finite, possibly empty, sequences of formulae, and A, B and C are arbitrary formulae.

(i) $\langle\langle\Gamma,A\rangle, \langle\Gamma,B\rangle, \langle\Gamma,A\wedge B\rangle\rangle \in \to\wedge$.

(ii) $\langle\langle\Gamma,A,C\rangle, \langle\Gamma,A\wedge B,C\rangle\rangle, \langle\langle\Gamma,B,C\rangle, \langle\Gamma,A\wedge B,C\rangle\rangle$
$\in \wedge\to$.

(iii) $\langle\langle\Gamma,A\rangle, \langle\Gamma,A\vee B\rangle\rangle, \langle\langle\Gamma,B\rangle, \langle\Gamma,A\vee B\rangle\rangle \in \to\vee$.

(iv) $\langle\langle\Gamma,A,C\rangle, \langle\Gamma,B,C\rangle, \langle\Gamma,A\vee B,C\rangle\rangle \in \vee\to$.

(v) $\langle\langle\Gamma,A,B\rangle, \langle\Gamma,A\supset B\rangle\rangle \in \to\supset$.

(vi) $\langle\langle\Gamma,A\rangle, \langle\Gamma,B,C\rangle, \langle\Gamma,A\supset B,C\rangle\rangle \in \supset\to$.

(vii) $\langle\langle\Gamma,A\rangle, \langle\Gamma,\forall\xi A_x(\xi)\rangle\rangle \in \to\forall$, provided $x \notin \Gamma$.

(viii) $\langle\langle\Gamma,A,C\rangle, \langle\Gamma,\forall\xi A_t(\xi),C\rangle\rangle \in \forall\to$.

(ix) $\langle\langle\Gamma,A\rangle, \langle\Gamma,\exists\xi A_t(\xi)\rangle\rangle \to\exists$.

(x) $\langle\langle\Gamma,A,C\rangle, \langle\Gamma,\exists\xi A_x(\xi),C\rangle\rangle \in \exists\to$, provided $x \notin \Gamma, C$

(xi) $\langle\langle\Gamma,C\rangle, \langle\Gamma,A,C\rangle\rangle \in \theta$.

(xii) $\langle\langle\Gamma,A,A,C\rangle, \langle\Gamma,A,C\rangle\rangle \in \kappa$.

(xiii) $\langle\langle\Gamma,A,\Delta,B,\Theta,C\rangle, \langle\Gamma,B,\Delta,A,\Theta,C\rangle\rangle \in \pi$.

The formulae denoted above by A, B, $A\wedge B$, $A\vee B$, $A\supset B$, $\forall\xi A_x(\xi)$ and $\exists\xi A_x(\xi)$ occuring in the conclusions of applications of the rules are called the <u>principle formulae</u>. Only the rule π has more than one principle formula in any application. The formulae A and B occuring in the premises are called the side formulae. The free variable denoted by x in (vii) and (x) is called the <u>eigenvariable</u> (of the application); the term denoted by t in (viii) and (ix) is called the <u>eigenterm</u>. Rules θ, κ and π called <u>structural</u>, the others are called <u>operational</u>. Of the rules to be introduced below (see 1.12, 1.13, 1,14 and 1.17) Nx, Fj and Px are operational, σ is structural.

1.12 Definition

<u>The system LD</u> is defined by:

$$\text{(i) } Ax(D) = Ax(M).$$

$$\text{(ii) } Rule(D) = Rule(M) \cup \{Nx\},$$

where Nx is the rule determined by

$$\langle\langle\Gamma, A, B\rangle, \langle\Gamma, \neg A, B\rangle, \langle\Gamma, B\rangle\rangle \in Nx,$$

Γ, A and B arbitrary.

1.13 Definition

<u>The system LJ</u> is defined by:

$$\text{(i) } Ax(J) = Ax(M).$$

$$\text{(ii) } Rule(J) = Rule(M) \in \{Fj\},$$

where Fj is the rule determined by

$$\langle\langle\Gamma, \perp\rangle, \langle\Gamma, A\rangle\rangle \in Fj,$$

Γ and A arbitrary.

1.14 Definition

<u>The system LE</u> is defined by:

$$\text{(i) } Ax(E) = Ax(M).$$

$$\text{(ii) Rule}(E) = \text{Rule}(M) \cup \{Px\},$$

where Px is the rule determined by

$$\langle\langle \Gamma, A \supset B, A\rangle, \langle \Gamma, A\rangle\rangle \in Px,$$

Γ, A and B arbitrary.

1.15 Definition

The system LK is defined by:

(i) Ax(K) = Ax(M).

$$\text{(ii) Rule}(K) = \text{Rule}(M) \cup \{Fj, Px\}.$$

1.16 Definitions

From this point on, unless otherwise specified, the letters LX will stand for an arbitrary one of the systems LM, LD, LJ, LE or LK.

A formula A is said to be provable in LX, provided the sequent $\longrightarrow A$ is derivable, in this case we write $\vdash_X A$. Let Γ be any set of formulae and let A be a formula, then A is said to be provable from the assumptions Γ in LX, written $\Gamma \vdash_X A$, provided there is some finite subset Γ_0 of Γ so that $\Gamma_0 \longrightarrow A$ is derivable (it being understood that Γ_0 is written down in some definite order as a sequence).

1.17 Theorem (Curry(63))

The cut rule σ determined by

$$\langle\langle \Gamma, A\rangle, \langle \Gamma, A, B\rangle, \langle \Gamma, B\rangle\rangle \in \sigma,$$

is a derived rule of each of the systems LX.

1.18 Theorem

The Deduction Theorem

For any of the systems LX, $\Gamma \cup \{G\} \vdash_X A$ iff $\Gamma \vdash_X G \supset A$, G and A arbitrary formulae and Γ an arbitrary set of formulae.

Proof. Obvious.

1.19 Theorem

Nx is a derived rule of LB.

Proof. Suppose we have a derivation $f = \langle T_1, p_1 \rangle$ with end-sequent $p_1(e_1) = \langle \Gamma, A, B \rangle$, and a derivation $g = \langle T_2, p_2 \rangle$ with end-sequent $p_2(e_2) = \langle \Gamma, \neg A, B \rangle$. We need to find a derivation of $\langle \Gamma, B \rangle$. For simplicity's sake we assume the sets of points S_1 and S_2 of T_1 and T_2 to be disjoint, and we put

$$S = S_1 \cup S_2 \cup \{y_1, \ldots, y_7, e\},$$

where the y_i and e are not elements of $S_1 \cup S_2$. We now define a function $s: S - \{e\} \longrightarrow S$ by

$$s(x) = \begin{cases} s_1(x) & \text{if } x \in S_1 - \{e_1\} \\ s_2(x) & \text{if } x \in S_2 - \{e_2\} \end{cases}$$

$s(y_i) = y_{i+1}$, $1 \leq i \leq 4$, $s(e_1) = y_2$, $s(e_2) = y_6$, $s(y_6) = y_7$, $s(y_7) = y_5$ and $s(y_5) = e$, where s_1 and s_2 are the successor functions of T_1 and T_2.

Let J_1 and J_2 be the sets of junction points of T_1 and T_2. Then we put

$$J = J_1 \cup J_2 \cup \{ y_2, y_5 \},$$

and define $\theta : J \longrightarrow S \times S$ by

$$\theta(x) = \begin{cases} \theta_1(x) & \text{if } x \in J_1 \\ \theta_2(x) & \text{if } x \in J_2 \end{cases},$$

and $\theta(y_2) = \langle e_1, y_1 \rangle$; $\theta(y_5) = \langle y_4, y_7 \rangle$.

Clearly, $T = \langle S, s, e, \theta \rangle$ is a tree.

Let $p : S \longrightarrow S_L$ be given by

$$p(x) = \begin{cases} p_1(x) & \text{if } x \in S_1 \\ p_2(x) & \text{if } x \in S_2 \end{cases},$$

$p(y_1) = \langle \Gamma, A, \bot, \bot \rangle$, $p(y_2) = \langle \Gamma, A, \neg B, \bot \rangle$, $p(y_3) = \langle \Gamma, \neg B, A, \bot \rangle$, $p(y_4) = \langle \Gamma, \neg B, \neg A \rangle$, $p(y_5) = \langle \Gamma, \neg B, B \rangle$, $p(y_6) = \langle \Gamma, \neg \bot, \neg B, B \rangle$, $p(y_7) = \langle \Gamma, \neg B, \neg A, B \rangle$ and $p(e) = \langle \Gamma, B \rangle$.

It is now easily verified that $h = \langle T, p \rangle$ is a derivation in LE with end-sequent $\Gamma \longrightarrow B$.

In future proofs will be written informally, making use of whatever obvious abbreviations are available. For example, the above derivation would appear as

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, A \xrightarrow{f} B \qquad \Gamma, A, \bot \longrightarrow \bot}{\Gamma, \neg B, A \longrightarrow \bot}\ {\supset}{\rightarrow}}{\Gamma, \neg B \longrightarrow \neg A}\ {\rightarrow}{\supset} \qquad \Gamma, \neg A \xrightarrow{g} B}{\Gamma, \neg B \longrightarrow B}\ \sigma}{\Gamma \longrightarrow B}\ \mathrm{Px}$$

where a double horizontal line takes the place of finitely many applications of θ, κ or π.

1.20 Definition

For the purposes of this definition we suppose that the underlying set V of the language L is well-ordered. (If this is not the case we use the axiom of choice to impose a well-order on V.)

Let all of the free variables appearing in a formula A be given by the list $x_1 = \langle 0, a_1 \rangle, \ldots, x_n = \langle 0, a_n \rangle$ determined by the well-ordering of V. Then the <u>closure of A</u> is defined to be the formula

$$\forall \xi_n \ldots \forall \xi_1 A_{x_1}(\xi_1) \ldots {}_{x_n}(\xi_n),$$

where $\xi_i = \langle 1, a_i \rangle$, $1 \leq i \leq n$. If A contains no free variables then the closure of A is A.

1.21 Definition

Let Σ be any set of sentences. We define the <u>theory $\mathcal{T}$ in the system LX generated by Σ</u> to be the ordered pair

$$\langle LX, \Sigma \rangle.$$

Elements of Σ are called the axioms of $\mathcal{T}$. A formula A is said to be a theorem of $\mathcal{T}$ iff $\Sigma \vdash_X A$.

1.22 Definition

A set of formulae Γ is said to be <u>consistent</u> provided $\Gamma \nvdash \bot$.

The empty set of formulae is easily seen to be consistent by consulting the definitions of the systems LX. This definition, which is more convenient than the usual one, is motivated by the following derivations which obtain in all systems.

(i) If Γ is inconsistent it contains a finite subset Γ_0 so that $\Gamma_0 \longrightarrow \perp$ is derivable. Let $G \in \Gamma_0$.

$$\dfrac{\Gamma_0 \longrightarrow G \qquad \dfrac{\dfrac{\Gamma_0 \longrightarrow \perp}{\Gamma_0, G \longrightarrow \perp}\,\theta}{\Gamma_0 \longrightarrow \neg G}\,\rightarrow\supset}{\Gamma_0 \longrightarrow G \wedge \neg G}\,\rightarrow\wedge$$

(ii) Suppose $\Gamma \vdash G \wedge \neg G$.

$$\dfrac{\Gamma_0 \longrightarrow G \wedge \neg G \qquad \dfrac{\dfrac{\Gamma_0, G \longrightarrow G \qquad \Gamma_0, \perp \longrightarrow \perp}{\Gamma_0, G, \neg G \longrightarrow \perp}\,\supset\rightarrow}{\Gamma_0, G \wedge \neg G \longrightarrow \perp}\,\wedge\rightarrow}{\Gamma_0 \longrightarrow \perp}\,\sigma$$

We also say that a theory $\mathcal{T}$ is <u>consistent</u> provided that $\perp$ is not a theorem of $\mathcal{T}$.

1.23 Remark

By theorem 1.19 it is clear that the following figure is valid provided we interpret $\mathrm{Der}(X) \longrightarrow \mathrm{Der}(Y)$ as meaning $\mathrm{Der}(Y) \subseteq \mathrm{Der}(X)$. The examples given in the accompanying table show that all inclusions are proper; a tick in a compartment indicates that the corresponding sequent is derivable in the indicated system, a cross that it is not derivable

Figure 1

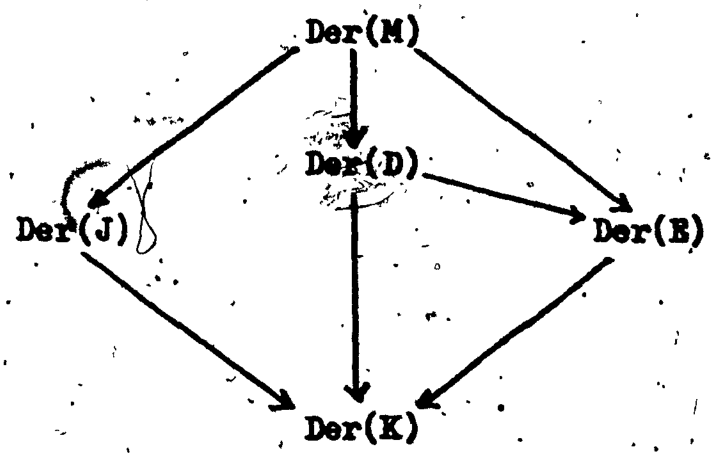

Table 1

	$\rightarrow A \vee \neg A$	$\bot \rightarrow A$	$(A \supset B) \supset A \rightarrow A$
LM	✗	✗	✗
LD	✓	✗	✗
LJ	✗	✓	✗
LE	✓	✗	✓
LK	✓	✓	✓

Section 2. Model Theory

2.1 Introduction

The first part of this section is devoted to the definition of the Lindenbaum algebras $\mathfrak{A}_X$ associated with the systems LX. Using the information developed we go on to define a model theory for each of the systems. For the most part our terminology concerning lattices is drawn from Rasiowa and Sikorski (70) and Curry (63). For convenience, we recall here that a distributive lattice $\mathfrak{A}$ is called

(i) <u>relatively pseudo-complemented</u> if for any a, b $\in \mathfrak{A}$, $\mathfrak{A}$ contains the relative pseudo-complement of a with respect to b

$$a \Rightarrow b = \bigwedge\{x\colon a \wedge x \leq b\};$$

(ii) <u>pseudo-Boolean</u> if it is relatively pseudo-complemented and has a zero element 0;

(iii) <u>classical</u> if for any a, b $\in \mathfrak{A}$

$$(a \Rightarrow b) \Rightarrow a = a.$$

Because of the peculiarities of our subject matter we shall, furthermore, adopt the terminology of the following definition.

2.2 Definition

A relatively pseudo-complemented lattice $\mathfrak{A}$ is said to be

of <u>type 1</u>, if $\mathfrak{A}$ contains a distinguished element $f \neq 1$;

of type 2, if $\mathfrak{A}$ is of type 1 and for each $a \in \mathfrak{A}$,

$$a \vee (a \Rightarrow f) = 1;$$

of type 3, if $\mathfrak{A}$ is of type 1 and $\mathfrak{A}$ is pseudo-Boolean with $f = 0$;

of type 4, if $\mathfrak{A}$ is of type 1 and classical;

of type 5, if $\mathfrak{A}$ is of type 1 and a Boolean algebra with $f = 0$.

2.3 Definition

For each system LX, we define a relation $\sim_X$ on F_L by

$$A \sim_X B \text{ iff } A \longrightarrow B,\ B \longrightarrow A \in \mathrm{Der}(X).$$

$\sim_X$ is an equivalence relation, for

(i) symmetry is part of the definition,

(ii) reflexivity is guaranteed by the definition of $\mathrm{Ax}(X)$,

and (iii) transitivity follows from theorem 1.17.

We also define

$$|A| = \{B \in F_L : A \sim_X B\},$$

and we say

$$|A| \leq |B| \text{ iff } A \longrightarrow B \in \mathrm{Der}(X).$$

It is easily seen that $\leq$ is a partial order.

The partially ordered set $\mathfrak{A}_X = \{|A| : A \in F_L\}$ is called the Lindenbaum algebra of LX. Note that if two systems LX and LY are equivalent (definition 1.10) then their Lindenbaum algebras are identical.

2.4 Theorem

$\mathfrak{A}_X$ is a relatively pseudo-complemented lattice with meet $\wedge$, join $\vee$, relative pseudo-complement $\Rightarrow$ and unit 1 in which

$$\text{(i)}\quad |A| \wedge |B| = |A \wedge B|,$$

$$\text{(ii)}\quad |A| \vee |B| = |A \vee B|,$$

$$\text{(iii)}\quad |A| \Rightarrow |B| = |A \supset B|,$$

and

$$\text{(iv)}\quad 1 = |A \supset A|,$$

where A and B are arbitrary formulae. Furthermore for each formula A we have

$$\text{(v)}\quad \bigwedge\{|A_X(t)| : t \in T_L\} = |\forall\xi A_X(\xi)|.$$

Proof. The proof is easy and we limit ourselves to demonstrating (v). We have for each $t \in T_L$

$$\frac{A_X(t) \longrightarrow A_X(t)}{\forall\xi A_X(\xi) \longrightarrow A_X(t)}$$

so that $|\forall\xi A_X(\xi)| \leqslant \bigwedge\{|A_X(t)| : t \in T_L\}$. On the other hand, suppose that $|B| \leqslant \bigwedge\{|A_X(t)| : t \in T_L\}$ and let y be a free variable not appearing in B. Then

$$B \longrightarrow A_X(y) \in \mathrm{Der}(X),$$

and so, by $\rightarrow\forall$,

$$|B| \leqslant |\forall\xi A_X(\xi)|.$$

2.5 Definition.

For each $a \in \mathfrak{A}_X$ we put $a^* = a \Rightarrow |\bot|$, and $\mathfrak{J}_X =$ $\{a : a^* = 1\}$.

By theorem 2.3 we have the following table corresponding to table 1 in which a tick in a compartment indicates that the corresponding statement is a theorem of the indicated lattice.

Table 2

	$a \vee a^* = 1$	$\mathcal{J}_X = \{\|\perp\|\}$	$(a \Rightarrow b) \Rightarrow a = a$
$\mathfrak{A}_M$			
$\mathfrak{A}_D$	✓		
$\mathfrak{A}_J$		✓	
$\mathfrak{A}_E$	✓		✓
$\mathfrak{A}_K$	✓	✓	✓

From the above it is clear that each lattice $\mathfrak{A}_X$ is of type 1 with $f = |\perp|$. Furthermore, $\mathfrak{A}_D$ is of type 2, $\mathfrak{A}_J$ is of type 3, $\mathfrak{A}_E$ is of type 4 and $\mathfrak{A}_K$ is of type 5.

As an analogue to figure 1 we have the following diagram, in which $\mathfrak{A}_X \longrightarrow \mathfrak{A}_Y$ is to be interpreted as the lattice morphism which assigns to each $|A| \in \mathfrak{A}_X$ the element $|A| \in \mathfrak{A}_Y$.

Figure 2

$$\begin{tikzcd}
 & \mathfrak{A}_M \arrow[dl] \arrow[d, no head] \arrow[dr] & \\
\mathfrak{A}_J \arrow[dr] & \mathfrak{A}_D \arrow[r] \arrow[d] & \mathfrak{A}_E \arrow[dl] \\
 & \mathfrak{A}_K &
\end{tikzcd}$$

The following theorems show that none of the Lindenbaum algebras are isomorphic, and that the empty spaces in table 2 can be filled with crosses.

2.6 Theorem

$$\mathcal{A}_M \not\cong \mathcal{A}_D.$$

Proof. Suppose that $\varphi : \mathcal{A}_M \longrightarrow \mathcal{A}_D$ is a lattice isomorphism. Let $|B|_M = \varphi^{-1}(|\perp|_D)$.

Since $|\perp|_D \neq 1$, it is clear that $|B|_M \neq 1$.

Let A be a formula for which $|A|_M \leqslant |B|_M$ and $|A|_M \neq 1$, and put $\varphi(|A|) = |A'|$. Then $|A \vee (A \supset B)|_M \neq 1$, but

$$\varphi(|A \vee (A \supset B)|) = |A'| \vee |A'|^* = 1.$$

Corollary.

$$\mathcal{A}_J \not\cong \mathcal{A}_D,\ \mathcal{A}_J \not\cong \mathcal{A}_E,\ \mathcal{A}_J \not\cong \mathcal{A}_K,\ \mathcal{A}_M \not\cong \mathcal{A}_E,\ \mathcal{A}_M \not\cong \mathcal{A}_K.$$

2.7 Theorem

$$\mathcal{A}_M \not\cong \mathcal{A}_J.$$

Proof. Suppose that $\varphi : \mathcal{A}_M \longrightarrow \mathcal{A}_J$ is a lattice isomorphism. Let $|A|_M = \varphi^{-1}(|\perp|) = \varphi^{-1}(0)$, and let B be any 0-ary relation symbol not appearing in A. By induction on A $|A \wedge B| < |A|$, and so

$$0 = 0 \wedge \varphi(|B|) = \varphi(|A \wedge B|) < \varphi(|A|) = 0.$$

Corollary

$$\mathcal{A}_E \not\cong \mathcal{A}_K.$$

2.8 Theorem

$$\mathfrak{A}_M \not\cong \mathfrak{A}_E.$$

Proof. Suppose that $\varphi : \mathfrak{A}_M \longrightarrow \mathfrak{A}_E$ is a lattice isomorphism. Let A and B be two distinct atomic formulae. Then

$$|(A \supset B) \supset A|_M \neq |A|_M,$$

but

$$\varphi(|(A \supset B) \supset A|) = \varphi(|A|).$$

Corollary

$$\mathfrak{A}_D \not\cong \mathfrak{A}_K.$$

2.9 Definition

A <u>realization R of LX in J and $\mathfrak{A}$</u> is an ordered triple

$$\langle R, J, \mathfrak{A} \rangle,$$

satisfying:

(i) J is a non-empty set, called the <u>domain of the realization</u>.

(ii) $\mathfrak{A}$ is a complete lattice of type 1 with distinguished element f. In addition,

if LX is LD then $\mathfrak{A}$ is of type 2,

if LX is LJ then $\mathfrak{A}$ is of type 3,

if LX is LE then $\mathfrak{A}$ is of type 4, and

if LX is LK then $\mathfrak{A}$ is of type 5.

(iii) R is a function on $C \cup (\bigcup_{n \in \omega} R^n) \cup F$ so that

(a) $R \upharpoonright C : C \longrightarrow J$;

(b) for each $n \in \omega$, $R \restriction R^n : R^n \longrightarrow \mathfrak{A}^{(J^n)}$, i.e. the value of R at B^n is a function $B^n{}_R : J^n \longrightarrow \mathfrak{A}$;

(c) $\perp_R = f$.

2.10 Definition

Let R be a realization of LX in J and $\mathfrak{A}$. Then for each $A \in F_L$ we shall define, inductively, a function

$$A_R : J^{FV} \longrightarrow \mathfrak{A}.$$

Elements of J^{FV} are called <u>valuations</u>. The definition follows. Let $v \in J^{FV}$.

(i) If $A = B^n(t_1, \ldots, t_n)$ $\quad At_L$ then

$$A_R(v) = B^n{}_R(j_1, \ldots, j_n),$$

where $j_i = v(t_i)$ if $t_i \in FV$, and $j_i = (t_i)_R$ if $t_i \in C$, $1 \leq i \leq n$.

(ii) If $A = B \wedge C$, $B \vee C$ or $B \supset C$, then $A_R(v) = B_R(v) \wedge C_R(v)$, $B_R(v) \vee C_R(v)$ or $B_R(v) \Rightarrow C_R(v)$, respectively.

(iii) If $A = \forall \xi\, B_x(\xi)$ or $\exists \xi\, B_x(\xi)$ then $A_R(v) = \bigwedge\{B_R(w_j) : j \in J\}$ or $\bigvee\{B_R(w_j) : j \in J\}$, respectively, where $w_j(y) = j$ if $y = x$, and $w_j(y) = v(y)$ otherwise.

2.11 Theorem

Let R be a realization of LX in J and $\mathfrak{A}$, and let A be any formula. If v and v' are valuations agreeing at each free variable appearing in A then $A_R(v) = A_R(v')$. In particular, if A is a sentence then A_R is constant.

Proof. By definition 2.10 this is obvious for A atomic, and the theorem clearly also holds for $A \wedge B$, $A \vee B$ and $A \supset B$ if it holds for A and for B. Let v and v' be valuations agreeing on all the free variables appearing in A except for x, and suppose that the theorem is true for A. Then

$$\begin{aligned} \forall \xi\, A_x(\xi)_R(v) &= \bigwedge\{A_R(w_j):\ j \in J\} \\ &= \bigwedge\{A_R(w'_j):\ j \in J\} \\ &= \forall \xi\, A_x(\xi)_R(v'). \end{aligned}$$

Similarly for $\exists \xi\, A_x(\xi)$.

2.12 Remark

The requirement in definition 2.9(ii) that $\mathcal{Q}$ be complete was made solely in order to ensure that the infinite suprema and infima of definition 2.10(iii) exist. In future we shall say that <u>R is a realization of LX in J and $\mathcal{Q}$</u>, even when $\mathcal{Q}$ is incomplete, provided definition 2.10(iii) makes sense. This modification allows the following definition.

2.13 Definition

The <u>canonical realization of LX</u> is the realization RX in T_L and $\mathcal{Q}_X$ determined by

(i) for each $c \in C$, $c_{RX} = c$;

(ii) for each $n \in \omega$, and each $B^n \in R^n$

$$B^n{}_{RX}(t_1, \ldots, t_n) = |B^n(t_1, \ldots, t_n)|;$$

(iii) $\perp_{RX} = |\perp|$.

For each $A \in F_L$ and each valuation $v: FV \longrightarrow T_L$, we denote by A^*_v the formula obtained from A by replacing every free variable x appearing in A by the term $v(x)$.

2.14 Theorem

For each $A \in F_L$ and each $v \in T_L^{FV}$

$$A_{RX}(v) = |A^*_v|.$$

Proof. The proof is an induction the length of A.

(1) Suppose $A = B^n(t_1,\ldots,t_n) \in At_L$. By definitions 2.10 and 2.13

$$A_{RX}(v) = B^n{}_R(t_1',\ldots,t_n'),$$

where $t_i' = t_i$ if $t_i \in C$, and $t_i' = v(t_i)$ otherwise. Hence, appealing again to 2.13,

$$A_{RX}(v) = |B^n(t_1',\ldots,t_n')| = |A^*_v|.$$

(2) If the theorem holds for formulae A and B, it clearly also holds for $A \wedge B$, $A \vee B$ and $A \supset B$.

Suppose that the theorem holds for $A = A_s(t)$, where $s,\ t \in T_L$. We then have

$$\begin{aligned} \forall\xi A_s(\xi)_{RX}(v) &= \bigwedge\{A_R(w_t) : t \in T_L\} \\ &= \bigwedge\{|A^*_{w_t}| : t \in T_L\} \\ &= \bigwedge\{|(A^*_v)_{v(s)}(t)| : t \in T_L\} \\ &= |\forall\xi(A^*_v)_{v(s)}(\xi)| \\ &= |(\forall\xi A_s(\xi))^*_v|, \end{aligned}$$

where w_t is defined in accordance with definition 2.10, and where the last equality but one holds by theorem 2.4. The case for $\exists\xi\, A_s(\xi)$ is handled similarly.

2.15 Definitions

Let $A \in F_L$, let $\Gamma, \Delta \subseteq F_L$, let R be a realization of LX in J and $\mathfrak{A}$, and let $v \in J^{FV}$.

(i) If $A_R(v) = 1$ we say that v satisfies A in R. If v satisfies each $A \in \Gamma$ in R, we say that v satisfies Γ in R.

(ii) If v satisfies A in R for each $v \in J^{FV}$, we say that R is a model for A, and we write

$$R \models_X A.$$

Similarly, we define R is a model for Γ, written

$$R \models_X \Gamma.$$

(iii) If each realization R of LX is a model for A, we say that A is valid ("X-valid", if the distinction is necessary), and we write

$$\models_X A.$$

If every element of Γ is valid we also write

$$\models_X \Gamma.$$

(iv) If, for each realization R, $R \models_X A$ whenever $R \models_X \Gamma$, we say Γ semantically entails A, and we write

$$\Gamma \models_X A.$$

If Γ semantically entails every element of Δ, then we also say that Γ semantically entails Δ, and we write

$$\Gamma \models_X \Delta.$$

(v) If Θ is a finite sequence of formulae $\langle T_1, \ldots, T_n \rangle$ then we shall write "$\bigwedge \Theta$" as an abbreviation for

$$(\ldots(T_1 \wedge T_2) \wedge \ldots) \wedge T_n.$$

2.16 Definitions

For each consistent set of sentences Σ we define a filter ∇_Σ in $\mathfrak{A}_X$ by

$$\nabla_\Sigma = \{|A| : \Sigma \vdash_X A\}.$$

For any $A, B \in F_L$ we define

(i) $A \approx B$ iff $|A| \Rightarrow |B|,\ |B| \Rightarrow |A| \in \nabla_\Sigma$;

(ii) $\|A\| = \{B \in F_L : A \approx B\}$;

(iii) $\|A\| \leqslant \|B\|$ iff $|A| \Rightarrow |B| \in \nabla_\Sigma$;

(iv) $\mathfrak{A}_X(\Sigma) = \{\|A\| : A \in F_L\}$;

(v) $h_\Sigma : \mathfrak{A}_X \longrightarrow \mathfrak{A}_X(\Sigma)$ is the mapping given by

$$|A| \longmapsto \|A\|.$$

2.17 Theorem

(i) $\approx$ is an equivalence relation on F_L;

(ii) If $|A| \leqslant |B|$ then $\|A\| \leqslant \|B\|$.

(iii) $\|A\| \wedge \|B\| = \|A \wedge B\|$, $\|A\| \vee \|B\| = \|A \vee B\|$, $\|A\| \Rightarrow \|B\| = \|A \supset B\|$, $\bigwedge\{\|A_x(t)\| : t \in T_L\} = \|\forall\xi A_x(\xi)\|$, and $\bigvee\{\|A_x(t)\| : t \in T_L\} = \|\exists\xi A_x(\xi)\|$.

(iv) $\|A\| = 1$ iff $|A| \in \nabla_\Sigma$.

(v) $\mathfrak{A}_X(\Sigma)$ is a lattice of type 1; if $X = D$ then $\mathfrak{A}_X(\Sigma)$ is of type 2, if $X = J$ then $\mathfrak{A}_X(\Sigma)$ is of type 3, if $X = E$ then $\mathfrak{A}_X(\Sigma)$ is of type 4 and if $X = K$ then $\mathfrak{A}_X$ is of type 5.

(vi) h_Σ is a lattice morphism.

Proof. The proofs for (i) and (ii) are trivial. It is easily seen that $\leqslant$ is a partial order on $\mathfrak{A}_X(\Sigma)$. (iii) follows easily, and we prove only the second to last equality as an example.

The derivation

$$\frac{A_X(t) \longrightarrow A_X(t)}{\forall\xi A_X(\xi) \longrightarrow A_X(t)}$$

shows $\|\forall\xi A_X(\xi)\| \le \bigwedge\{\|A_X(t)\|;\ t \in T_L\}$. On the other hand, if $\|B\| \le \|A_X(t)\|$ for some $B \in F_L$ and each $t \in T_L$, then we have

$$\Sigma_0, B \longrightarrow A_X(t)$$

derivable, for Σ_0 some finite subset of Σ and t a free variable not occuring in B. But then

$$\Sigma_0, B \longrightarrow \forall\xi A_X(\xi)$$

is derivable by an application of $\rightarrow\forall$, and we are done.

The proof of (iv) is again easy, and (v) and (vi) follow immediately from (iii).

2.18 Definitions

$\mathfrak{A}_X(\Sigma)$ is called the <u>Lindenbaum algebra of the theory $\mathcal{T} = \langle LX, \Sigma \rangle$</u>.

Let Σ be a consistent set of sentences. Then the <u>canonical realization $RX(\Sigma)$ of $\mathcal{T} = \langle LX, \Sigma \rangle$</u> is the realization in T_L and $\mathfrak{A}_X(\Sigma)$ defined by

(i) for each $c \in C$, $c_{RX(\Sigma)} = c$;

(ii) for each $n \in \omega$ and each $B^n \in R^n$, $B^n_{RX(\Sigma)} = h_\Sigma \circ B^n_{RX}$;

(iii) $\perp_{RX(\Sigma)} = \|\perp\|$,

2.19 Theorem

For any consistent set of sentences Σ, any $A \in F_L$ and any $v: FV \longrightarrow T_L$

$$A_{RX(\Sigma)}(v) = \|A_v^*\|.$$

Proof. Since h_Σ is a lattice homomorphism, the theorem is a corollary to theorem 2.14.

2.20 Theorem

If $\Gamma \longrightarrow A$ is derivable in LM, then for any realization R of LM

$$R \models_M (\bigwedge\Gamma) \supset A.$$

Proof. By induction on the derivation of $\Gamma \longrightarrow A$.
The theorem clearly holds if $\Gamma \longrightarrow A$ is an axiom.

If the theorem holds for the premises it clearly also holds for the conclusion of any of the rules θ, π, κ, $\wedge\rightarrow$, $\rightarrow\wedge$, $\vee\rightarrow$, $\rightarrow\vee$, $\supset\rightarrow$, $\rightarrow\exists$ or $\forall\rightarrow$.

Suppose, then, that $\Gamma \longrightarrow A$ is derived by $\rightarrow\supset$ from the premise $\Gamma, B \longrightarrow C$. But then if the theorem holds for the premise it holds also for the conclusion because of the identity

$$(a \wedge b) \Rightarrow c = a \Rightarrow (b \Rightarrow c),$$

which holds in any relatively pseudo-complemented lattice (Rasiowa and Sikorski (70), page 60).

Suppose $\Gamma \longrightarrow A$ is derived by $\rightarrow\forall$ from the premise $\Gamma \longrightarrow B(x)$. A then has the form $\forall\xi B_x(\xi)$. Let R be any realization in J and $\mathfrak{A}$, and let $v: FV \longrightarrow J$ be an arbitrary valuation. Since x does not appear in we have

$$(\bigwedge\Gamma)_R(v) = (\bigwedge\Gamma)_R(w_j) \leq B(x)_R(w_j)$$

for each $j \in J$, by the induction hypothesis, where w_j is defined as in definition 2.10. From this it is immediate that $R \models_M (\bigwedge\Gamma) \supset A$.

The case in which $\Gamma \longrightarrow A$ is derived by $\exists\!\rightarrow$ is handled analogously.

2.21 Theorem

If $\Gamma \longrightarrow A$ is derivable in LX, then for any realization R of LX

$$R \models_X (\bigwedge\Gamma) \supset A.$$

Proof. It suffices to contimue the induction of theorem 2.20 by considering the following three cases.

(1) $\Gamma \longrightarrow A$ is derived in LD by Nx from the premises $\Gamma, B \longrightarrow A$ and $\Gamma, \neg B \longrightarrow A$. But then if the theorem holds for the premises, it holds also for the conclusion since

$$((g \wedge b) \Rightarrow a) \wedge ((g \wedge (b \Rightarrow f)) \Rightarrow a) = (g \wedge (b \vee (b \Rightarrow f))) \Rightarrow a$$
$$= g \Rightarrow a,$$

in any lattice of type 2.

(2) $\Gamma \longrightarrow A$ is derived in LJ by Fj from the premise $\Gamma \longrightarrow \perp$. But then if the theorem holds for the premise, it holds also for the conclusion, since in any lattice $\mathfrak{A}$ of type 3

$$g \leq f \text{ iff } g \leq a, \text{ for each } a \in \mathfrak{A}.$$

(3) $\Gamma \longrightarrow A$ is derived in LE by Px from the premise $\Gamma, A \supset B \longrightarrow A$. But then if the theorem holds for the premise, it holds also for the conclusion since in any lattice of type 4

$$(g \wedge (a \Rightarrow b)) \Rightarrow a = g \Rightarrow ((a \Rightarrow b) \Rightarrow a)$$
$$= g \Rightarrow a.$$

2.22 Theorem

<u>The Correctness Theorem</u>

If $\Gamma \subseteq F_L$ and $\Gamma \vdash_X A$, then $\Gamma \models_X A$.

Proof. If $\Gamma \vdash_X A$, then by definition there is a finite subset Γ_0 of Γ for which $\Gamma_0 \longrightarrow A$ is derivable. Let R be any realization which is a model for Γ, and let v be an arbitrary valuation. Then $(\bigwedge \Gamma_0)_R(v) = 1$, but by the preceding theorem $(\bigwedge \Gamma_0)_R(v) \leq A_R(v)$.

2.23 Theorem

Let Σ be a consistent set of sentences. Then $RX(\Sigma) \models_X A$ whenever $\Sigma \vdash_X A$, for any formula A.

Proof. Suppose $\Sigma \vdash_X A$. By theorem 2.22 $\Sigma \models_X A$. Hence by definition 2.15 it is only necessary to show that $RX(\Sigma) \models_X \Sigma$.

Let $S \in \Sigma$, and let $v \in T_L^{FV}$. Then certainly $S \in \nabla_X$ (cf. definition 2.16, this would not necessarily be true if S were a formula) and so by theorems 2.17 and 2.19

$$S_{RX(\Sigma)}(v) = \|S_v^*\| = \|S\| = 1,$$

where the middle equality holds since S is a sentence.

2.24 Theorem

Let Σ be a consistent set of sentences. If $RX(\Sigma) \models_X A$ then $\Sigma \vdash_X A$, for any formula A.

Proof. Suppose $RX(\Sigma) \models_X A$, and let $v \in T_L^{FV}$. By theorems 2.17 and 2.19, and definition 2.16

$$1 = A_{RX(\Sigma)}(v) = \|A_v^*\|,$$

and so $|A_v^*| \in \nabla_\Sigma$. In particular, when v is the identity valuation (defined by $v(x) \doteq x$) we have

$$|A| = |A_v^*| \in \nabla_\Sigma,$$

and so by definition 2.16 $\Sigma \vdash_X A$.

2.25 Theorem

<u>The Completeness Theorem</u>

Let Σ be a consistent set of sentences. Then $\Sigma \vdash_X A$ whenever $\Sigma \models_X A$, for any formula A.

Proof. Suppose $\Sigma \models_X A$. From the proof of theorem 2.23 we know that $RX(\Sigma) \models_X \Sigma$, and so by definition 2.15 (iv) $RX(\Sigma) \models_X A$. Theorem 2.24 now gives the desired result.

2.26 Theorem

A set of formulae Σ is consistent iff every finite subset of Σ is consistent.

Proof. Obvious.

2.27 Theorem

The Compactness Theorem

A set of sentences Σ has a model iff every finite subset of Σ has a model.

Proof. By 2.26, it suffices to show that Σ has a model iff Σ is consistent. If Σ is consistent then $RX(\Sigma)$ is a model for Σ by theorem 2.23.

On the other hand, suppose Σ is inconsistent, and let R be a model for Σ. Then there is some finite subset Σ_0 of Σ so that $(\wedge\Sigma_0)_R(v) \leq \perp_R$, for each valuation v, (theorem 2.21). But $(\wedge\Sigma_0)_R(v) = 1$, so that $\perp_R = 1$ contradicting definition 2.9(iii).

Section 3. Craig's Interpolation Theorem

3.1 Introduction

We intend to adapt Schütte's proof of the theorem for the intuitionist predicate calculus (Schütte (62)). In order to do so we set up a correspondence between formulae in the conclusion of an application of a rule and the formulae in the premise or premises:

(i) The formulae in the conclusion denoted by Γ, Δ, Θ or C in definitions 1.11, 1.12, 1.13, 1.14 and 1.17 correspond to the same formulae in the premises.

(ii) The principle formula corresponds to the side formulae. In the case of the structural rules, this will be understood to mean that A and B in the conclusion of an application of π correspond to A and B in the premise, respectively; A in the conclusion of θ corresponds to nothing; and, A in the conclusion of κ corresponds to both A's in the premise.

If Δ is a sequence of formulae occurring in a conclusion, we shall denote the corresponding sequence of formulae in the premise (or premises) by Δ_0, Δ_1, etc. The method of proof is proof-theoretic; a model-theoretic proof will be found in Gabbay (71).

3.2 Definition

For any formula A, $[A]$ will denote the set of relation symbols and free variables appearing in A. For any sequence Γ of formulae $\langle G_1, \ldots, G_m \rangle$ we also define

$$[\Gamma] = [G_1] \cup \ldots \cup [G_m],$$

and
$$[\Gamma, A] = [\Gamma] \cup [A].$$

If Γ and Δ are finite sequences of formulae, Γ_Δ will denote the subsequence of Γ obtained by deleting from Γ each formula occurring in Δ.

3.3 Theorem

<u>Craig's Interpolation Theorem</u>

Let $\Gamma \longrightarrow D \in Der(M)$ and let $\Delta \subseteq \Gamma$. Then either $\Gamma_\Delta \longrightarrow D \in Der(M)$, or there is some formula J satisfying

(a) $[J] \subseteq [\Delta] \cap [\Gamma_\Delta, D]$,

and (b) $\Delta \longrightarrow J$ and $J, \Gamma_\Delta \longrightarrow D$ are derivable.

A formula J satisfying (a) and (b) is called an <u>interpolation formula for $\Gamma \longrightarrow D$ and Δ</u>.

Proof. If Δ is empty the theorem holds trivially, hence we shall always assume Δ is non-empty.

The proof will be an induction on the derivation of $\Gamma \longrightarrow D$.

(1) $\Gamma \longrightarrow D$ is an axiom. If $D \in \Delta$ then (a) and (b) are satisfied for $J = D$, otherwise $D \in \Gamma_\Delta$ and so $\Gamma_\Delta \longrightarrow D$ is derivable.

(2) $\Gamma \longrightarrow D$ is the conclusion of an application of one of

the rules $\wedge\rightarrow$, $\rightarrow\vee$, $\rightarrow\supset$, π, θ or κ, with premise

$$\Gamma_1 \longrightarrow D_1 .$$

Let Δ_1 be the subsequence of Γ_1 which corresponds to Δ.

If $\Gamma_{1\Delta_1} \longrightarrow D_1$ is derivable, then either $\Gamma_\Delta \longrightarrow D$ is $\Gamma_{1\Delta_1} \longrightarrow D_1$ or is derivable from it by a single application of the rule in question. Otherwise there is, by the induction hypothesis, an interpolation formula J for $\Gamma_1 \longrightarrow D_1$ and Δ_1. From the structure of the rules it is clear that (b) also holds for $\Gamma \longrightarrow D$ and Δ using the same J, and since

$$[\Delta_1] \cap [\Gamma_{1\Delta_1}, D_1] \subseteq [\Delta] \cap [\Gamma_\Delta, D],$$

(a) is satisfied as well.

(3) $\Gamma \longrightarrow D$ is the conclusion of an application of one of the rules $\rightarrow\wedge$, $\vee\rightarrow$ or $\supset\rightarrow$, with premises

$$\Gamma_1 \longrightarrow D_1 \text{ and } \Gamma_2 \longrightarrow D_2 .$$

Let Δ_1 and Δ_2 be the subsequences of Γ_1 and Γ_2 corresponding to Δ.

There are three subcases to consider: (i) the principle formula of the conclusion does not occur in Δ; (ii) the principle formula occurs in Δ, and the rule is $\vee\rightarrow$; (iii) the principle formula occurs in Δ, and the rule is $\supset\rightarrow$.

(i) We note that $\Delta = \Delta_1 = \Delta_2$. If $\Gamma_{1\Delta} \longrightarrow D_1$ and $\Gamma_{2\Delta} \longrightarrow D_2$ are derivable, then so is $\Gamma_\Delta \longrightarrow D$ by a single application of the rule in question.

If $\Gamma_{1\Delta} \longrightarrow D_1$ is derivable and there is an interpolation formula J for $\Gamma_2 \longrightarrow D_2$ and Δ, then J is also an interpolation formula for $\Gamma \longrightarrow D$ and Δ. Similarly, if $\Gamma_{2\Delta} \longrightarrow D_2$ is derivable and there is an interpolation formula J for $\Gamma_1 \longrightarrow D_1$ and Δ, then J is also an interpolation formula for $\Gamma \longrightarrow D$ and Δ.

If, finally, there is an interpolation formula J_1 for $\Gamma_1 \longrightarrow D_1$ and Δ, and an interpolation formula J_2 for $\Gamma_2 \longrightarrow D_2$ and Δ, then $J_1 \wedge J_2$ is an interpolation formula for $\Gamma \longrightarrow D$ and Δ.

(ii) $\Gamma \longrightarrow D$ has the form $\Phi, A \vee B \longrightarrow D$ with premises

$$\Phi, A \longrightarrow D \text{ and } \Phi, B \longrightarrow D,$$

and Δ has the form $\Psi, A \vee B$.

It is clear that if either of $\Gamma_{1\Delta_1} \longrightarrow D_1$ and $\Gamma_{2\Delta_2} \longrightarrow D_2$ is derivable, then so is $\Gamma_\Delta \longrightarrow D$. Otherwise there are interpolation formulae J_1 and J_2 so that

$(\Psi, A =)\ \Delta_1 \longrightarrow J_1$ and $(\Psi, B =)\ \Delta_2 \longrightarrow J_2$,

and also

$$J_1, \Phi_\Psi \longrightarrow D \text{ and } J_2, \Phi_\Psi \longrightarrow D,$$

are derivable. From this it is clear that $J_1 \vee J_2$ satisfies (b). But

$$[J_1 \vee J_2] \subseteq \{[\Psi, A] \cap [\Phi_\Psi, D]\} \cup \{[\Psi, B] \cap [\Phi_\Psi, D]\}$$
$$= [\Psi, A \vee B] \cap [\Phi_\Psi, D]$$
$$= [\Delta] \cap [\Gamma_\Delta, D],$$

and (a) is satisfied as well. Thus $J_1 \vee J_2$ is an interpolation formula for $\Gamma \longrightarrow D$ and Δ.

(iii) $\Gamma \longrightarrow D$ has the form $\Psi, A \supset B \longrightarrow D$ with premises

$$\Phi \longrightarrow A \text{ and } \Phi, B \longrightarrow D,$$

and Δ has the form $\Psi, A \supset B$ where Ψ is a subsequence of Φ.

Applying the induction hypothesis to $\Phi, B \longrightarrow D$ and Ψ, B we know that either $\Phi_{\Psi} \longrightarrow D$ is derivable or there is an interpolation formula J_2 for $\Phi, B \longrightarrow D$ and Ψ, B. If the former possibility obtains then it is immediate that $\Gamma_{\Delta} \longrightarrow D$ is derivable, hence we suppose that the latter situation holds.

We next apply the induction hypothesis to $\Phi \longrightarrow A$ and Φ_{Ψ}. If $\Psi \longrightarrow A$ is derivable we are done because of the derivation

$$\frac{\Psi \longrightarrow A \qquad \Psi, B \longrightarrow J_2}{\Psi, A \supset B \longrightarrow J_2}$$

Otherwise there is an interpolation formula J_1 for $\Phi \longrightarrow A$ and Φ_{Ψ}. Thus

$$\Phi_{\Psi} \longrightarrow J_1 \text{ and } J_1, \Psi \longrightarrow A$$

are both derivable, and it is clear that $J_1 \supset J_2$ satisfies (b) of the theorem. But

$$[J_1] \subseteq [\Phi_{\Psi}] \cap [\Psi, A] \subseteq [\Gamma_{\Delta}] \cap [\Delta],$$

and

$$[J_2] \subseteq [\Psi, B] \cap [D] \subseteq [\Delta] \cap [D].$$

Whence

$$[J_1 \supset J_2] \subseteq [\Delta] \cap ([\Gamma_{\Delta}] \cup [D]),$$

and (a) is satisfied as well.

(4) $\Gamma \longrightarrow D$ is the conclusion of an application of one of the rules $\forall\rightarrow$, $\rightarrow\forall$, $\exists\rightarrow$ or $\rightarrow\exists$, with premise $\Gamma_1 \longrightarrow D_1$.

Now $\Gamma_\Delta \longrightarrow D$ is $\Gamma_{1\Delta_1} \longrightarrow D_1$ or is derivable from it by a single application of the rule in question. Hence if $\Gamma_{1\Delta_1} \longrightarrow D_1$ is derivable we are done; otherwise there is an interpolation formula J satisfying (a) and (b) for $\Gamma_1 \longrightarrow D_1$ and Δ_1.

(i) The principle formula does not occur in Δ, and $\Delta_1 = \Delta$.

(α) The rule is $\forall\rightarrow$ or $\rightarrow\exists$. There is no restriction on variables, and we may derive

$$\Delta \longrightarrow J \text{ and } J, \Gamma_\Delta \longrightarrow D.$$

Either we are done or the eigenterm t of the application is a free variable occurring in J but not in $[\Gamma_\Delta, D]$. In this case we may derive

$$\frac{\Delta \longrightarrow J}{\Delta \longrightarrow \exists\xi J_t(\xi)} \text{ and } \frac{J, \Gamma_\Delta \longrightarrow D}{\exists\xi J_t(\xi), \Gamma_\Delta \longrightarrow D}$$

and $\exists\xi J_t(\xi)$ is an interpolation formula for $\Gamma \longrightarrow D$ and Δ.

(β) The rule is $\rightarrow\forall$ or $\exists\rightarrow$. Since the restriction on variables is satisfied for the original derivation, the eigenvariable x of the application does not occur in Δ, and hence also does not occur in J. It follows that J is an interpolation formula for $\Gamma \longrightarrow D$ and Δ.

(ii) The principle formula occurs in Δ, $\Gamma_{1\Delta_1} = \Gamma_\Delta$, and the rule in question is either $\forall\to$ or $\exists\to$.

(α) The rule is $\forall\to$. There is no restriction on variables and we may derive

$$\Delta \longrightarrow J \text{ and } J, \Gamma_\Delta \longrightarrow D$$

by the same rule. Either we are done or the eigenterm t of the application is a free variable occurring in J but not in Δ. In this case we may derive

$$\frac{\Delta \longrightarrow J}{\Delta \longrightarrow \forall\xi J_t(\xi)} \text{ and } \frac{J, \Gamma_\Delta \longrightarrow D}{\forall\xi J_t(\xi), \Gamma_\Delta \longrightarrow D}$$

and $\forall\xi J_t(\xi)$ is an interpolation formula for $\Gamma \longrightarrow D$ and Δ.

(β) The rule is $\exists\to$. Since the restriction on variables is satisfied for the original derivation, the eigenvariable x of the application does not occur in Γ_Δ or D, and hence does not occur in J. It follows that J is an interpolation formula for $\Gamma \longrightarrow D$ and Δ.

3.4 Remark

In order to prove Craig's theorem for the propositional fragment of LD, we introduce a new system LD_1, obtained from LD by replacing the rule Nx by a rule Nx_1, for which the theorem may be easily established. It is easily shown that Nx_1 is a derived rule of LD, so that any sequent derivable in LD_1 is also derivable in LD. Thus the two systems will have been shown to be equivalent if we can show that Nx is a derived rule of LD_1. The latter result is again easy provided that the cut rule σ is a derived rule of LD_1 (Curry (63) claims that this is so but does not provide a proof, see the footnote on page 266). The theorem we have used to establish this point is theorem 3.7, a rediscovery of an early result due to Curry (Curry (52c)). Unfortunately, theorem 3.7 is valid only for the propositional fragment of LD_1, bringing about the state of affairs mentioned in the introduction to the thesis.

3.5 Definitions

<u>The system LD_1</u> is defined by:

$$\text{(i) } Ax(D_1) = Ax(M).$$

$$\text{(ii) } Rule(D_1) = Rule(M) \cup \{Nx_1\},$$

where Nx_1 is the rule determined by

$$<<\Gamma,\neg A,A>,<\Gamma,A>> \in Nx_1,$$

and A arbitrary.

<u>The system LD_1*</u> is defined by:

$$\text{(i) } Ax(D_1{*}) = Ax(M).$$

$$\text{(ii) } Rule(D_1{*}) = Rule(D_1) \cup \{\sigma\}.$$

3.6 Theorem

Craig's theorem holds for LD_1

Proof. It suffices to continue the induction of theorem 3.3 to the case in which $\Gamma \longrightarrow D$ is derived by Nx_1. By the induction hypothesis, either

$$\Gamma_{\Delta}, \neg D \longrightarrow D$$

is derivable, in which case by Nx_1 so also is $\Gamma_{\Delta} \longrightarrow D$, or there is an interpolation formula J for the premise and Δ. Since

$$[\Gamma_{\Delta}, \neg D, D] = [\Gamma_{\Delta}, D],$$

it is evident that J is also an interpolation formula for the conclusion and Δ.

3.7 Theorem

Any derivation in the propositional fragment of LD_1* can be transformed into a derivation having the same end-sequent with at most a single application of Nx_1, and so that this application of Nx_1 is the last inference made in the derivation.

Proof. It is sufficient to consider derivations with the properties:

(i) At least one of the premises of the last inference is the conclusion of an application of the rule Nx_1.

(ii) The derivation involves no other applications of Nx_1 except, possibly, in the final inference.

Once the theorem has been established for derivations satisfying the above conditions, a trivial induction extends the theorem to arbitrary derivations.

We examine in turn the eleven different cases which can arise depending on what rule legitimizes the last inference.

(1) $\wedge\rightarrow$. The end of the derivation then has the form

$$\dfrac{\dfrac{\Gamma, B, \neg A \longrightarrow A}{\Gamma, B \longrightarrow A}\ Nx_1}{\Gamma, B \wedge C \longrightarrow A}\ \wedge\rightarrow$$

This is transformed to

$$\dfrac{\dfrac{\underline{\Gamma, B, \neg A \longrightarrow A}}{\Gamma, B \wedge C, \neg A \longrightarrow A}}{\Gamma, B \wedge C \longrightarrow A}\ Nx_1$$

(2) $\rightarrow\wedge$. The end of the derivation then has the form

$$\dfrac{\dfrac{\Gamma, \neg A \longrightarrow A}{\Gamma \longrightarrow A}\ Nx_1 \qquad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ \rightarrow\wedge$$

if the left premise is derived by Nx_1. This is transformed to

$$\dfrac{\dfrac{\dfrac{\dfrac{\underline{\dfrac{\dfrac{\dfrac{}{\Gamma, A \longrightarrow A} \qquad \dfrac{\Gamma \longrightarrow B}{\Gamma, A \longrightarrow B}\ \theta}{\Gamma, A \longrightarrow A \wedge B}\ \rightarrow\wedge \qquad \Gamma, A, \bot \longrightarrow \bot}{\Gamma, A, \neg(A \wedge B) \longrightarrow \bot}\ \supset\rightarrow}}{\Gamma, \neg(A \wedge B) \longrightarrow \neg A}\ \rightarrow\supset \qquad \Gamma, \neg A \longrightarrow A}{\underline{\Gamma, \neg(A \wedge B) \longrightarrow A}} \qquad \dfrac{\Gamma \longrightarrow B}{\Gamma, \neg(A \wedge B) \longrightarrow B}\ \theta}{\Gamma, \neg(A \wedge B) \longrightarrow A \wedge B}\ \rightarrow\wedge}{\Gamma \longrightarrow A \wedge B}\ Nx_1$$

If the right premise is derived by Nx_1 the transformation is similar. If both premises are derived by Nx_1 the end of the derivation has the form

$$\dfrac{\dfrac{\Gamma, \neg A \longrightarrow A}{\Gamma \longrightarrow A}\,Nx_1 \qquad \dfrac{\Gamma, \neg B \longrightarrow B}{\Gamma \longrightarrow B}\,Nx_1}{\Gamma \longrightarrow A \wedge B}\,{\to}\wedge$$

This is transformed to

$$\dfrac{\dfrac{\dfrac{\dfrac{(\alpha) \qquad (\beta)}{\Gamma, B, \neg(A\wedge B) \longrightarrow A\wedge B}\,\sigma \qquad \bot \longrightarrow \bot}{\Gamma, B, \neg(A\wedge B) \longrightarrow \bot}\,{\supset}{\to}}{\Gamma, \neg(A\wedge B) \longrightarrow \neg B}\,{\to}{\supset} \qquad \dfrac{(\gamma) \qquad (\delta)}{\Gamma, \neg(A\wedge B), \neg B \longrightarrow A\wedge B}\,\sigma}{\dfrac{\Gamma, \neg(A\wedge B) \longrightarrow A\wedge B}{\Gamma \longrightarrow A\wedge B}\,Nx_1}\,\sigma$$

where (α) stands for

$$\dfrac{\dfrac{\dfrac{A \longrightarrow A \qquad B \longrightarrow B}{\Gamma, B, A \longrightarrow A\wedge B}\,{\to}\wedge \qquad \bot \longrightarrow \bot}{\Gamma, B, A, \neg(A\wedge B) \longrightarrow \bot}\,{\supset}{\to}}{\Gamma, B, \neg(A\wedge B) \longrightarrow \neg A}\,{\to}{\supset}$$

(β) stands for

$$\dfrac{\Gamma, \neg A \longrightarrow A \qquad B \longrightarrow B}{\Gamma, B, \neg(A\wedge B), \neg A \longrightarrow A\wedge B}\,{\to}\wedge$$

(γ) stands for

$$\dfrac{\dfrac{\dfrac{A \longrightarrow A \qquad \Gamma, \neg B \longrightarrow B}{\Gamma, \neg B, A \longrightarrow A\wedge B}\,{\to}\wedge \qquad \bot \longrightarrow \bot}{\Gamma, \neg B, A, \neg(A\wedge B) \longrightarrow \bot}\,{\supset}{\to}}{\Gamma, \neg(A\wedge B), \neg B \longrightarrow \neg A}\,{\to}{\supset}$$

and (δ) stands for

$$\dfrac{\Gamma, \neg A \longrightarrow A \qquad \Gamma, \neg B \longrightarrow B}{\Gamma, \neg(A\wedge B), \neg B, \neg A \longrightarrow A\wedge B}\,{\to}\wedge$$

(3) $\vee\!\rightarrow$. The end of the derivation then has the form

$$\dfrac{\dfrac{\Gamma, B, \neg A \longrightarrow A}{\Gamma, B \longrightarrow A}\,Nx_1 \qquad \Gamma, C \longrightarrow A}{\Gamma, B \vee C \longrightarrow A}\,\vee\!\rightarrow$$

This is transformed to

$$\begin{array}{cl} \Gamma, B, \neg A \longrightarrow A \qquad \Gamma, C \longrightarrow A & \\ \hline\hline \Gamma, B \vee C, \neg A \longrightarrow A & \vee\!\rightarrow \\ \hline \Gamma, B \vee C \longrightarrow A & Nx_1 \end{array}$$

The case in which both premises are derived by Nx_1 is clearly no more complicated.

(4) $\rightarrow\!\vee$. The end of the derivation then has the form

$$\dfrac{\dfrac{\Gamma, \neg A \longrightarrow A}{\Gamma \longrightarrow A}\,Nx_1}{\Gamma \longrightarrow A \vee B}\,\rightarrow\!\vee$$

This is transformed to

$$\begin{array}{cl} A \longrightarrow A & \\ \hline A \longrightarrow A \vee B \qquad \bot \longrightarrow \bot & \rightarrow\!\vee \\ \hline\hline A, \neg(A \vee B) \longrightarrow \bot & \supset\!\rightarrow \\ \hline\hline \neg(A \vee B) \longrightarrow \neg A \qquad \Gamma, \neg A \longrightarrow A & \rightarrow\!\supset \\ \hline\hline \Gamma, \neg(A \vee B) \longrightarrow A & \\ \hline \Gamma, \neg(A \vee B) \longrightarrow A \vee B & \rightarrow\!\vee \\ \hline \Gamma \longrightarrow A \vee B & Nx_1 \end{array}$$

(5) $\supset\!\rightarrow$. If the left premise is derived by Nx_1 the end of the derivation has the form

$$\dfrac{\dfrac{\Gamma, \neg A \longrightarrow A}{\Gamma \longrightarrow A}\,Nx_1 \qquad \Gamma, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C}\,\supset\!\rightarrow$$

This is transformed to

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \longrightarrow A \quad \Gamma, B \longrightarrow C}{\Gamma, A \supset B, A \longrightarrow C}{\supset\rightarrow} \qquad \bot \longrightarrow \bot}{\Gamma, A \supset B, A, \neg C \longrightarrow \bot}{\supset\rightarrow}}{\Gamma, A \supset B, \neg C \longrightarrow \neg A}{\rightarrow\supset} \qquad \Gamma, \neg A \longrightarrow A}{\Gamma, A \supset B, \neg C \longrightarrow A}{\sigma} \qquad \Gamma, B \longrightarrow C}{\dfrac{\Gamma, A \supset B, \neg C \longrightarrow C}{\Gamma, A \supset B \longrightarrow C}\,Nx_1}\,{\supset\rightarrow}
$$

If the right premise is derived by Nx_1 the end of the derivation has the form

$$
\dfrac{\Gamma \longrightarrow A \qquad \dfrac{\Gamma, B, \neg C \longrightarrow C}{\Gamma, B \longrightarrow C}\,Nx_1}{\Gamma, A \supset B \longrightarrow C}\,{\supset\rightarrow}
$$

This is transformed to

$$
\dfrac{\dfrac{\Gamma \longrightarrow A \qquad \Gamma, B, \neg C \longrightarrow C}{\Gamma, A \supset B, \neg C \longrightarrow C}\,{\supset\rightarrow}}{\Gamma, A \supset B \longrightarrow C}\,Nx_1
$$

If both premises are derived by Nx_1 the end of the derivation has the form

$$
\dfrac{\dfrac{\Gamma, \neg A \longrightarrow A}{\Gamma \longrightarrow A}\,Nx_1 \qquad \dfrac{\Gamma, B, \neg C \longrightarrow C}{\Gamma, B \longrightarrow C}\,Nx_1}{\Gamma, A \supset B \longrightarrow C}\,{\supset\rightarrow}
$$

This is transformed to

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \longrightarrow A \qquad \Gamma, B, \neg C \longrightarrow C}{\Gamma, A \supset B, A, \neg C \longrightarrow C}\,{\supset\to} \qquad \bot \longrightarrow \bot}{\Gamma, A \supset B, A, \neg C \longrightarrow \bot}\,{\supset\to}}{\Gamma, A \supset B, \neg C \longrightarrow \neg A}\,{\to\supset} \qquad \Gamma, \neg A \longrightarrow A}{\Gamma, A \supset B, \neg C \longrightarrow A}\,\sigma \qquad \Gamma, B, \neg C \longrightarrow C}{\Gamma, A \supset B, \neg C \longrightarrow C}\,{\supset\to}}{\Gamma, A \supset B \longrightarrow C}\,\mathrm{Nx}_1$$

(6) $\to\supset$. The end of the derivation then has the form

$$\dfrac{\dfrac{\Gamma, B, \neg A \longrightarrow A}{\Gamma, B \longrightarrow A}\,\mathrm{Nx}_1}{\Gamma \longrightarrow B \supset A}\,{\to\supset}$$

This is transformed to

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A, B \longrightarrow A}{A \longrightarrow B \supset A}\,{\to\supset} \qquad \bot \longrightarrow \bot}{A, \neg(B \supset A) \longrightarrow \bot}\,{\supset\to}}{\neg(B \supset A) \longrightarrow \neg A}\,{\to\supset} \qquad \Gamma, B, \neg A \longrightarrow A}{\Gamma, \neg(B \supset A), B \longrightarrow A}\,\sigma}{\Gamma, \neg(B \supset A) \longrightarrow B \supset A}\,{\to\supset}}{\Gamma \longrightarrow B \supset A}\,\mathrm{Nx}_1$$

(7) Nx_1. The end of the derivation then has the form

$$\dfrac{\dfrac{\Gamma, \neg A, \neg A \longrightarrow A}{\Gamma, \neg A \longrightarrow A}\,\mathrm{Nx}_1}{\Gamma \longrightarrow A}\,\mathrm{Nx}_1$$

Clearly the first application of Nx_1 may be replaced by an application of κ.

(8) θ, (9) κ, (10) π. For these cases the transformations are trivial.

(11) σ. If the left premise is derived by Nx_1 the end of the derivation has the form

$$\dfrac{\dfrac{\Gamma, \neg A \longrightarrow A}{\Gamma \longrightarrow A}\,Nx_1 \qquad \Gamma, A \longrightarrow B}{\Gamma \longrightarrow B}\,\sigma$$

This is transformed to

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, A \longrightarrow B \quad \perp \longrightarrow \perp}{\Gamma, A, \neg B \longrightarrow \perp}\,{\supset}{\rightarrow}}{\Gamma, \neg B \longrightarrow \neg A}\,{\rightarrow}{\supset} \qquad \dfrac{\Gamma, \neg A \longrightarrow A \quad \Gamma, A \longrightarrow B}{\Gamma, \neg A \longrightarrow B}\,\sigma}{\Gamma, \neg B \longrightarrow B}\,\sigma}{\Gamma \longrightarrow B}\,Nx_1$$

If the right premise is derived by Nx_1 the transformation is

$$\dfrac{\dfrac{\Gamma \longrightarrow A \quad \Gamma, A, \neg B \longrightarrow B}{\Gamma, \neg B \longrightarrow B}\,\sigma}{\Gamma \longrightarrow B}\,Nx_1$$

If both premises are derived by Nx_1, the transformation is

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, A, \neg B \longrightarrow B \quad \perp \longrightarrow \perp}{\Gamma, A, \neg B \longrightarrow \perp}\,{\supset}{\rightarrow}}{\Gamma, \neg B \longrightarrow \neg A}\,{\rightarrow}{\supset} \qquad \dfrac{\Gamma, \neg A \longrightarrow A \quad \Gamma, A, \neg B \longrightarrow B}{\Gamma, \neg B, \neg A \longrightarrow B}\,\sigma}{\Gamma, \neg B \longrightarrow B}\,\sigma}{\Gamma \longrightarrow B}\,Nx_1$$

3.8 Theorem

The propositional fragments of the systems LD_1 and LD_1* are equivalent. I.e. σ is a derived rule of the propositional fragment of LD_1.

Note: In the proof of this theorem, as well as in theorems 3.9 and 3.10, the letters LD, LD_1 and LD_1* will denote only the propositional fragments of the systems defined in 1.12, and 3.5.

Proof. Let $\Gamma \longrightarrow A$ be derivable in LD_1*, and let $\Gamma \xrightarrow{f} A$ be the derivation given by theorem 3.7. If f involves no applications of Nx_1, then, since σ is a derived rule of LM, $f \in Der(M) \subseteq Der(D_1)$. If f does involve an application of Nx_1 then it has a sub-derivation $\Gamma, \neg A \xrightarrow{g} A \in Der(M) \subseteq Der(D_1)$; but then obviously $f \in Der(D_1)$ as well. In any case $\Gamma \longrightarrow A$ is derivable in LD_1, and σ is a derived rule of LD_1.

3.9 Theorem

The systems LD and LD_1 are equivalent.

Proof. Nx_1 is a derived rule of LD by

$$\frac{\Gamma, A \longrightarrow A \qquad \Gamma, \neg A \longrightarrow A}{\Gamma \longrightarrow A}\, Nx$$

Nx is a derived rule of LD_1* by

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, A \longrightarrow A \vee \neg A \qquad \bot \longrightarrow \bot}{\Gamma, A, \neg(A \vee \neg A) \longrightarrow \bot}\,\supset\rightarrow}{\Gamma, \neg(A \vee \neg A) \longrightarrow \neg A}\,\rightarrow\supset}{\Gamma, \neg(A \vee \neg A) \longrightarrow A \vee \neg A}\,\rightarrow\vee}{\Gamma \longrightarrow A \vee \neg A}\,Nx_1 \qquad \dfrac{\Gamma, A \longrightarrow B \qquad \Gamma, \neg A \longrightarrow B}{\Gamma, A \vee \neg A \longrightarrow B}\,\vee\rightarrow}{\Gamma \longrightarrow B}$$

But then by theorem 3.8 Nx is also a derived rule of LD_1.

3.10 Theorem

Craig's theorem holds for LD.

Proof. Theorems 3.6 and 3.9.

3.11 Theorem

Craig's theorem holds for LJ.

Proof. It suffices to continue the induction of theorem 3.3 to the case in which $\Gamma \longrightarrow D$ is derived by Fj. By the induction hypothesis, either

is derivable, in which case by Fj so also is $\Gamma_\Delta \longrightarrow D$, or there is an interpolation formula for the premise and Δ. Since

$$[\Delta] \cap [\Gamma_\Delta, \perp] \subseteq [\Delta] \cap [\Gamma_\Delta, D]$$

it is evident that the same formula is an interpolation formula for the conclusion and Δ.

3.12 Remark

The difficulty in extending the proof of Craig's theorem to LE resides in the non-constructive nature of the rule Px: given a derivation of $\Gamma, A \supset B \longrightarrow A$ and $\Delta \subseteq \Gamma$ there does not seem to be any obvious way of finding an interpolation formula J not involving predicates and variables occurring in B. In order to get around this difficulty we introduce a new system LE_m which is equivalent to LE (in the sense of theorem 3.14 below), and to which, since it does not possess the rule Px, the proof for LM can be extended.

3.13 Definition

The language $L_m = \langle A_{L_m}, T_{L_m}, F_{L_m}, S_{L_m} \rangle$ is defined by:

(i) $A_{L_m} = A_L$.

(ii) $T_{L_m} = T_L$.

(iii) $F_{L_m} = F_L$.

(iv) $S_{L_m} = \{\langle \Gamma, \Theta \rangle : \Gamma, \Theta$ finite, possibly empty, sequences of formulae$\}$

An element $\langle \Gamma, \Theta \rangle$ of S_{L_m} will be written informally as $\Gamma \longrightarrow \Theta$.

The system $LE_m = \langle L_m, Ax(E_m), Rule(E_m), Der(E_m) \rangle$ is defined by:

(i) $Ax(E_m) = Ax(M)$.

(ii) $Rule(E_m)$ is obtained from $Rule(M)$ by replacing C by Θ in $\wedge\rightarrow$, $\vee\rightarrow$, $\supset\rightarrow$, Θ, κ and π, by inserting Θ following the side and principle formulae in $\rightarrow\wedge$, $\rightarrow\vee$ and $\rightarrow\supset$, and by adding, in the obvious way, three new rules $\rightarrow\Theta$, $\rightarrow\kappa$ and $\rightarrow\pi$.

(iii) $Der(E_m)$ is defined analogously to $Der(M)$.

We quote the following theorem from Curry (63).

3.14 Theorem

Let Γ and Θ be finite sequences of formulae, and let C be the formula obtained from Θ by replacing all commas by disjunction symbols. Then $\Gamma \longrightarrow \Theta$ is derivable in LE_m iff $\Gamma \longrightarrow C$ is derivable in LE.

3.15 Theorem

Let $\Gamma \longrightarrow \Theta \in Der(E_m)$ and let $\Delta \subseteq \Gamma$. Then either $\Gamma_\Delta \longrightarrow \Theta \in Der(E_m)$ or there is some formula J satisfying

(a) $[J] \subseteq [\Delta] \cap [\Gamma_\Delta, \Theta]$,

and (b) $\Delta \longrightarrow J$ and $J, \Gamma_\Delta \longrightarrow \Theta$ are derivable.

Proof. The proof is an induction on the derivation of $\Gamma \longrightarrow \Theta$, adapted trivially from the proof of theorem 3.3 for LM.

3.16 Theorem

Craig's theorem holds for LE.

Proof. Theorems 3.14 and 3.15.

3.17 Theorem

Craig's theorem holds for LK.

Proof. Theorems 3.11 and 3.16. (Alternatively, see Kleene (67)).

Section 4. Categorical Representations

4.1 Introduction

In this section we present a categorical interpretation of propositional fragments of each of the systems LM, LD, LJ and LE (the reason for the exclusion of LK will be given below), and we show that when suitably interpreted, Craig's theorem holds in the interpretations. The definition of the categories $\underline{\mathcal{X}}_X$ corresponding to the fragments of the systems LX is based on one form of the categorical interpretation developed by Szabo and Lambek with respect to the system LJ. The interpretations, inter alia, enforce equivalence relations on derivations which allow a systematic comparison of derivations of the same sequent. The interest of finding a categorical interpretation is also suggested by the following points.

(a) The Lindenbaum algebras $\mathfrak{A}_X$ provide an extremely convenient means of characterizing the structure of the different systems. This characterization is so convenient that we have been able to rely on it almost exclusively in in developing the model theory of the systems LX. It suffers, however, from a twofold disadvantage. In the first place there is no distinction made between identity and interderivability of formulae; and in the second place, all information concerning the derivation of a given derivable sequent is discarded. The success of the Lindenbaum algebras suggests the great potential utility of finding an algebraic characterization of the systems which does not suffer from these latter drawbacks.

(b) Any category may be construed as a sequent calculus in which the formulae are the objects of the category, the derivations are the morphisms, and the only rule of inference is the operation of composition of morphisms. This fact makes it natural to try to reverse the procedure and interpret sequent calculi as categories.

For further motivation and details concerning (b) see, for example, Lambek (a). We next establish some conventions concerning terminology by means of the following definition.

4.2 Definition

For simplicity's sake we restrict our attention to the categorical fragments of the systems LX. In addition, in order to facilitate the interpretation, and in order to simplify the categorical proofs of Craig's theorem we establish the following conventions which will remain in force for the remainder of the thesis.

(i) We add a new symbol $\top$ to the language L, and we specify $\top \in At_L$.

(ii) For each system LX, Ax(X) is redefined as

$$Ax(X) = \{A \longrightarrow A : A \in F_L\} \cup \{\longrightarrow \top\}.$$

(iii) The rule σ is taken in the form

$$\frac{\Gamma \longrightarrow A \qquad \Delta, A, \Theta \longrightarrow C}{\Delta, \Gamma, \Theta \longrightarrow C}$$

(iv) The rule π is taken in the form

$$\frac{\Gamma, A, B, \Delta \longrightarrow C}{\Gamma, B, A, \Delta \longrightarrow C}$$

(v) From now on the letters LM will denote the propositional fragment of the system defined in 1.11 subject to the modifications above.

(vi) From now on the letters LD will denote the propositional fragment of the system LD_1 defined in 3.5 subject to the modifications above.

(vii) From now on the letters LJ will denote the propositional fragment of the system defined in 1.13 subject to the modifications above.

(viii) From now on the letters LE will denote the propositional fragment of the system LE_m defined in 3.1 subject to the modifications above.

4.3 Remark

The definitions of the categories $\mathcal{X}_X$ are given in paragraphs 4.4 to 4.7 below. These definitions are adapted directly from the papers of M.E. Szabo, and no attempt is made here to justify the selection of the particular categories given. However, the search for a suitable categorical interpretation was guided by certain objectives which we state here informally, partly as a substitute for such justification, and partly as a convenient means of introducing new terminology and notation.

(a) The objects of $\mathcal{X}_X$, the category corresponding to LX, shall be the formulas of LX, and, as far as possible, the morphisms of $\mathcal{K}_X$ shall be the derivations of LX.

(b) For any formula A, the identity morphism on A, 1_A, shall be the derivation

$$A \longrightarrow A.$$

(c) Suppose we have derivations $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$, with corresponding morphisms $A \xrightarrow{\hat{f}} B$ and $B \xrightarrow{\hat{g}} C$. Then the composition of $\hat{f}$ and $\hat{g}$ shall be the morphism corresponding to the derivation

$$\frac{A \xrightarrow{f} B \qquad B \xrightarrow{g} C}{A \longrightarrow C}\,\sigma$$

(d) Suppose we have derivations $\Gamma \xrightarrow{f} A$ and $\Gamma \xrightarrow{g} A$, and derivations f' and g' given by

$$\frac{\Gamma \xrightarrow{f} A}{\Delta \longrightarrow B}\,r \quad \text{and} \quad \frac{\Gamma \xrightarrow{g} A}{\Delta \longrightarrow B}\,r$$

respectively, where r is some binary rule. If $\hat{f} = \hat{g}$ then we wish to have also $\hat{f'} = \hat{g'}$. Similarly for ternary rules.

(e) For each system LX, the corresponding category $\mathcal{K}_X$ shall be a category of the same sort as the corresponding Lindenbaum algebra $\mathfrak{A}_X$.

The qualification "as far as possible" in (a) is necessitated by the fact that, as derivations, the following are distinct

$$\frac{\dfrac{A \xrightarrow{f} B \qquad B \xrightarrow{g} C}{A \longrightarrow C}\,\sigma \qquad C \xrightarrow{h} D}{A \longrightarrow D}\,\sigma \qquad\qquad \frac{A \xrightarrow{f} B \qquad \dfrac{B \xrightarrow{g} C \qquad C \xrightarrow{h} D}{B \longrightarrow D}\,\sigma}{A \longrightarrow D}\,\sigma$$

However, if they are to be construed as morphisms, they must be identified in order to obtain associativity of composition (objective (c)). This makes it clear that morphisms of $\mathscr{X}_X$ at best correspond to equivalence classes of derivations, for some suitable equivalence relation ≡.

In order to assign a categorical meaning to a derivation of $\Gamma \longrightarrow A$ when Γ contains more than one formula we specify that the object of $\mathscr{X}_X$ corresponding to Γ shall be the formula $\bigwedge\Gamma$ (see definition 2.15(v)). We also agree that the object corresponding to an empty antecedent of a sequent shall be the formula $\top$. This accounts for the introduction of the symbol into the language L. (We note that from a proof-theoretical point of view this convention is harmless, since $\longrightarrow A$ is derivable if and only if $\top \longrightarrow A$ is derivable.)

The fact that the morphisms of $\mathscr{X}_X$ will prove to be equivalence classes of derivations explains why we have chosen to work with the systems LD_1 and LE_m rather than with LD and LE. To prove Craig's theorem we shall have to show that certain diagrams commute, and in order to do this we require explicit information concerning the derivations of $\Delta \longrightarrow J$ and $J, \Gamma_\Delta \longrightarrow D$. But theorems 3.10 and 3.16 provide no explicit information and we are forced to fall back on the proof-theoretically equivalent auxiliary systems LD_1 and LE_m.

We also note here that in the category $\mathscr{X}_K$, generated from the system LK in the same manner as the categories

$\underline{\mathcal{X}}_K$ below, for any two objects A and B [A,B] (the set of morphisms with domain A and co-domain B) is either a singleton or empty (see Szabo (a)). Under these circumstances Craig's theorem "lifts" trivially to $\underline{\mathcal{X}}_K$. This explains our omission of LK from consideration.

4.4 Definition

Let $\underline{At}_L$ be the discrete category generated by At_L. We define $\underline{\mathcal{X}}_M$ to be the free cartesian closed category with finite, non-empty sums generated by $\underline{At}_L$, with underlying category $\mathcal{X}_M$:

$$\underline{\mathcal{X}}_M = \langle \mathcal{X}_M, \top, \wedge, \vee, \supset \rangle,$$

where

(i) $\top$ is a terminal object in $\mathcal{X}_M$.

(ii) $\wedge : \mathcal{X}_M \times \mathcal{X}_M \longrightarrow \mathcal{X}_M$ is a product bi-functor.

(iii) $\vee : \mathcal{X}_M \times \mathcal{X}_M \longrightarrow \mathcal{X}_M$ is a sum bi-functor.

(iv) $\supset : \mathcal{X}_M^{op} \times \mathcal{X}_M \longrightarrow \mathcal{X}_M$ is an internal hom functor.

For any objects A, B, C there are thus natural isomorphisms

$$\Pi_{(A,B,C)} : [A, B \wedge C] \xrightarrow{\sim} [A,B] \times [A,C],$$
$$\Sigma_{(A,B,C)} : [A \vee B, C] \xrightarrow{\sim} [A,C] \times [B,C],$$
$$\Omega_{(A,B,C)} : [A \wedge B, C] \xrightarrow{\sim} [B, A \supset C],$$
$$\Psi_A : [A, \top] \xrightarrow{\sim} \{*\}.$$

In particular for any objects A, B, C we have coherent, natural isomorphisms

$$\alpha_{(A,B,C)} : A \wedge (B \wedge C) \longrightarrow (A \wedge B) \wedge C,$$
$$\gamma_{(A,B,C)} : A \vee (B \vee C) \longrightarrow (A \vee B) \vee C,$$
$$\zeta_{(A,B,C)} : A \wedge (B \vee C) \longrightarrow (A \wedge B) \vee (A \wedge C),$$

$$\nu_{(A,B)}: A\wedge B \xrightarrow{\sim} B\wedge A,$$

$$\mu_{(A,B)}: A\vee B \xrightarrow{\sim} B\vee A,$$

$$\lambda_A: A \xrightarrow{\sim} A\wedge T.$$

In future we shall omit subscripts on morphisms and transformations whenever the context makes it clear what these should be.

In addition we define for any objects A and B, morphisms δ_A, $\pi_{1(A\wedge B)}$, $\pi_{2(A\wedge B)}$, β_A, $\sigma_{1(A\vee B)}$, $\sigma_{2(A\vee B)}$, τ_A, $\varepsilon_{(A,B)}$ and $\eta_{(A,B)}$, by

$$\delta_A = \Pi^{-1}(1_A, 1_A),$$

$$\Pi(1_{A\wedge B}) = \langle \pi_1, \pi_2 \rangle,$$

$$\beta_A = \Sigma^{-1}(1_A, 1_A),$$

$$\Sigma(1_{A\vee B}) = \langle \sigma_1, \sigma_2 \rangle,$$

$$\tau_A \in [A, T],$$

$$\varepsilon_{(A,B)} = \Omega^{-1}(1_{A\supset B}),$$

and

$$\eta_{(A,B)} = \Omega(\nu_{(B,A)}).$$

We are now in a position to set up a "dictionary" for the intended interpretation of morphisms as equivalence classes of derivations.

LM	$\mathcal{L}_M$
$A \longrightarrow A$	1_A
$\dfrac{\Gamma, A \xrightarrow{f} C}{\Gamma, A\wedge B \longrightarrow C}\ \wedge\!\rightarrow$	$\hat{f}\circ(1\wedge\pi_1)$
$\dfrac{\Gamma, B \xrightarrow{f} C}{\Gamma, A\wedge B \longrightarrow C}\ \wedge\!\rightarrow$	$\hat{f}\circ(1\wedge\pi_2)$
$\dfrac{\Gamma \xrightarrow{f} A \quad \Gamma \xrightarrow{g} B}{\Gamma \longrightarrow A\wedge B}\ \rightarrow\!\wedge$	$(\hat{f}\wedge\hat{g})\circ\delta$

$$\frac{\Gamma \xrightarrow{f} A}{\Gamma \longrightarrow A \vee B} \to\vee \qquad \sigma_1 \circ \hat{f}$$

$$\frac{\Gamma \xrightarrow{f} B}{\Gamma \longrightarrow A \vee B} \to\vee \qquad \sigma_2 \circ \hat{f}$$

$$\frac{\Gamma, A \xrightarrow{f} C \qquad \Gamma, B \xrightarrow{g} C}{\Gamma, A \vee B \longrightarrow C} \vee\to \qquad \beta \circ (\hat{f} \vee \hat{g}) \circ \zeta$$

$$\frac{\Gamma \xrightarrow{f} A \qquad \Gamma, B \xrightarrow{g} C}{\Gamma, A \supset B \longrightarrow C} \supset\to \qquad \hat{g} \circ (1 \wedge \varepsilon) \circ (1 \wedge \hat{f} \wedge 1) \circ (\delta \wedge 1)$$

$$\frac{\Gamma, A \xrightarrow{f} B}{\Gamma \longrightarrow A \supset B} \to\supset \qquad \Omega(\hat{f} \circ \nu)$$

$$\frac{\Gamma \xrightarrow{f} C}{\Gamma, A \longrightarrow C} \Theta \qquad \hat{f} \circ \pi_1$$

$$\frac{\Gamma, A, A \xrightarrow{f} C}{\Gamma, A \longrightarrow C} \varkappa \qquad \hat{f} \circ (1 \wedge \delta)$$

$$\frac{\Gamma, A, B, \Delta \xrightarrow{f} C}{\Gamma, B, A, \Delta \longrightarrow C} \pi \qquad \hat{f} \circ (1 \wedge \nu \wedge 1)$$

$$\frac{\Gamma \xrightarrow{f} B \qquad \Delta, B, \Theta \xrightarrow{g} C}{\Delta, \Gamma, \Theta \longrightarrow C} \sigma \qquad \hat{g} \circ (1 \wedge \hat{f} \wedge 1)$$

(Because of coherence, applications of α have been ignored.)

4.5 Definition

The category $\underline{\mathcal{K}}_D$ is defined exactly as $\underline{\mathcal{K}}_M$ except that we postulate additional isomorphisms

$$\Theta_{(A,B)} : [A, \neg B \supset B] \xrightarrow{\sim} [A, B],$$

natural in A and B, and we define $\rho_A : \neg A \supset A \longrightarrow A$ by

$$\rho_A = \Theta(1_{\neg A \supset A}).$$

The "dictionary" of 4.3 is now expanded to include

LD	$\mathcal{L}_D$
$\dfrac{\Gamma, \neg A \xrightarrow{f} A}{\Gamma \longrightarrow A}\ Nx_1$	$\sim \rho_A \circ \Omega(\hat{f} \cdot \hat{\nu})$

4.6 Definition

The category $\mathcal{L}_J$ is defined exactly as $\mathcal{L}_M$ except that we postulate in addition that $\perp$ be an initial object. We thus have a bijection

$$\Phi_A : [\perp, A] \longrightarrow \{*\},$$

for each object A of $\mathcal{L}_J$. We define $\iota_A : \perp \longrightarrow A$ by

$$\iota_A \in [\perp, A],$$

and expand the "dictionary" of 4.3 to include

LJ	$\mathcal{L}_J$
$\dfrac{\Gamma \xrightarrow{f} \perp}{\Gamma \longrightarrow A}\ Fj$	$\iota_A \circ \hat{f}$

4.7 Definition

The category $\mathcal{L}_E$ is defined exactly as $\mathcal{L}_M$ except that we postulate in addition isomorphisms

$$\Delta_{(A,B,C)} : [A \wedge (B \supset C), B] \xrightarrow{\sim} [A, B],$$

for any objects A, B and C in $\mathcal{L}_E$.

In order to assign a categorical meaning to a derivation of $\Gamma \longrightarrow \Theta$ when Θ contains more than one formula we specify that the object of $\mathcal{L}_E$ corresponding to $\Theta = \langle T_1, \ldots, T_n \rangle$ shall be the formula $\bigvee\Theta$ defined by

$$(\ldots(T_1 \vee T_2) \vee \ldots) \vee T_n.$$

With the exception of $\rightarrow\supset$ the obvious modifications are to be made to the "dictionary" of 4.3. For eaxmple, the interpretation of

$$\frac{\Gamma \xrightarrow{f} \Theta, A \qquad \Gamma, B \xrightarrow{g} \Theta}{\Gamma, A \supset B \longrightarrow \Theta}$$

is the morphism

$$\beta \circ (\pi_2 \vee \hat{g}) \circ \zeta \circ [1 \wedge ((\pi_1 \vee \varepsilon) \circ \zeta \circ (f \wedge 1))] \circ (\delta \wedge 1).$$

The interpretation of

$$\frac{\Gamma, A \xrightarrow{f} \Theta, B}{\Gamma \longrightarrow \Theta, A \supset B}$$

is the monstrosity

$$\Delta(\sigma_{2((\vee\Theta)\vee(A\supset B))} \circ \Omega(\pi_{2([A\wedge(\wedge\Gamma)]\wedge B)} \circ (1 \wedge \varepsilon) \circ [1 \wedge (\beta \circ [\sigma_{1((\vee\Theta)\vee(A\supset B))} \vee (\sigma_{2((\vee\Theta)\vee(A\supset B))} \circ \Omega(\pi_{2(A\wedge B)}))] \circ \hat{f} \circ \nu_{(A,(\wedge\Gamma))}) \wedge 1] \circ (\delta_{A\wedge(\wedge\Gamma) \wedge 1[(\vee\Theta)\vee(A\supset B)]\supset B}))).$$

This is best understood by consulting the following derivation (which is valid in the system LB of definition 1.14), where $\Gamma, A \xrightarrow{f'} (\vee\Theta) \vee B$ is the derivation given by theorem 3.14.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, A \xrightarrow{f'} (\vee\Theta)\vee B \qquad \dfrac{\dfrac{\vee\Theta \longrightarrow \vee\Theta}{\vee\Theta \to (\vee\Theta)\vee(A\supset B)}\,{\scriptstyle \to\vee} \qquad \dfrac{\dfrac{\dfrac{B \to B}{B, A \to B}\,{\scriptstyle \theta}}{B \to A\supset B}\,{\scriptstyle \to\supset}}{B \to (\vee\Theta)\vee(A\supset B)}\,{\scriptstyle \to\vee}}{(\vee\Theta)\vee B \to (\vee\Theta)\vee(A\supset B)}\,{\scriptstyle \vee\to}}{\Gamma, A \longrightarrow (\vee\Theta)\vee(A\supset B)}\,{\scriptstyle \sigma} \qquad \dfrac{B \to B}{\Gamma, A, B \to B}\,{\scriptstyle \theta}}{\Gamma, A, ((\vee\Theta)\vee(A\supset B))\supset B \to B}\,{\scriptstyle \supset\to}}{\dfrac{\Gamma, ((\vee\Theta)\vee(A\supset B))\supset B \to A\supset B}{\Gamma, ((\vee\Theta)\vee(A\supset B))\supset B \to (\vee\Theta)\vee(A\supset B)}\,{\scriptstyle \to\vee}}\,{\scriptstyle \to\supset}$$

$$\dfrac{\Gamma, ((\vee\Theta)\vee(A\supset B))\supset B \to (\vee\Theta)\vee(A\supset B)}{\Gamma \longrightarrow (\vee\Theta)\vee(A\supset B)}\,\mathrm{Px}$$

The modified "dictionary" is now expanded to include

LB	$\mathcal{L}_B$
$\dfrac{\Gamma \xrightarrow{f} \Theta}{\Gamma \longrightarrow \Theta, A} \to \theta$	$\sigma_1 \circ f$
$\dfrac{\Gamma \xrightarrow{f} \Theta, A, A}{\Gamma \longrightarrow \Theta, A} \to \kappa$	$(1 \vee \beta) \circ f$
$\dfrac{\Gamma \xrightarrow{f} \Theta, A, B, \Delta}{\Gamma \longrightarrow \Theta, B, A\ \Delta} \to \pi,$	$(1 \wedge \mu \wedge 1) \circ f$

4.8 Theorem

Craig's theorem (3.3, 3.6, 3.11 and 3.15) holds for each of the modified systems LX in the following form:

Let $\Gamma \longrightarrow D$ be derivable and let $\Delta \subseteq \Gamma$, then there exists a formula J and derivations g and h satisfying

$$\text{(a) } [J] \subseteq [\Delta] \cap [\Gamma_\Delta, D],$$

and $$\text{(b) } \Delta \xrightarrow{g} J \text{ and } J, \Gamma_\Delta \xrightarrow{h} D.$$

Proof. If $\Gamma_\Delta \longrightarrow D$ is not derivable, the proof is a trivial modification of the proofs given above for the full systems. If $\Gamma_\Delta \longrightarrow D$ is derivable then so are both of

$$\Delta \longrightarrow \top \text{ and } \top, \Gamma_\Delta \longrightarrow D$$

and $\top$ is an interpolation formula for $\Gamma \longrightarrow D$ and Δ.

4.9 Theorem

Craig's theorem is valid for $\mathcal{L}_M$ in the sense that if J is an interpolation formula for $\Gamma \xrightarrow{f} D$ and Δ, and if $\Delta \xrightarrow{g} J$ and $J, \Gamma_\Delta \xrightarrow{h} D$ are the derivations determined by theorem 4.8, then in $\underline{\mathcal{K}}_M$ the following diagram commutes

$$\begin{tikzcd}
\bigwedge\Gamma \arrow[rr, "\hat{f}"] \arrow[d, dashed] & & D \\
(\bigwedge\Delta)\wedge(\bigwedge\Gamma_\Delta) \arrow[rr, "\hat{g}\wedge 1"] & & J\wedge(\bigwedge\Gamma_\Delta) \arrow[u, "\hat{h}"']
\end{tikzcd}$$

where the broken arrow denotes the isomorphism determined by the coherence of α and ν. In particular, when $\Gamma = \Delta$ we have

$$\begin{tikzcd}
\bigwedge\Gamma \arrow[rr, "\hat{f}"] \arrow[dr, "\hat{g}"'] & {} \arrow[d, phantom, "="] & D \\
& J \arrow[ur, "\hat{h}"'] &
\end{tikzcd}$$

Proof. The proof is an induction on the construction of the derivations g and h.

(1) If Δ is void the interpolation formula is $\top$, and the theorem reduces to the claim that the diagram

$$\begin{tikzcd}
\bigwedge\Gamma \arrow[rr, "\hat{f}"] \arrow[d, "\lambda^{-1}"'] & & D \\
\top\wedge(\bigwedge\Gamma) \arrow[rr, "1\wedge 1"] & & \top\wedge(\bigwedge\Gamma) \arrow[u, "\hat{h}=\hat{f}\cdot\lambda"']
\end{tikzcd}$$

commutes. Since this is always trivially true, we shall always suppose that $\Delta \neq 0$.

(2) If $\Gamma \longrightarrow D$ is an axiom it is of the form $A \longrightarrow A$, and for Δ non-void we have A as an interpolation formula clearly satisfying the theorem.

(3) It is easily seen from the proof of theorem 3.3 that the proof of the induction step reduces to showing that the following pairs of derivations are equivalent (i.e. are interpreted as the same morphism in $\underline{\mathcal{E}}_M$).

(i)

$$\dfrac{\dfrac{\Delta \xrightarrow{g} J \qquad J,\Phi \xrightarrow{h} C}{\Delta,\Phi \longrightarrow C}\,\sigma}{\Delta,\Phi,A \longrightarrow C}\,\theta \qquad\qquad \begin{array}{c} \dfrac{\dfrac{\Delta \xrightarrow{g} J}{\Delta,A \longrightarrow J}\,\theta \qquad J,\Phi \xrightarrow{h} C}{\Delta,A,\Phi \longrightarrow C}\,\sigma \\ \hline\hline \Delta,\Phi,A \longrightarrow C \end{array}$$

(ii)

$$\dfrac{\dfrac{\Delta \xrightarrow{g} J \qquad J,\Phi \xrightarrow{h} C}{\Delta,\Phi \longrightarrow C}\,\sigma}{\Delta,\Phi,A \longrightarrow C}\,\theta \qquad\qquad \dfrac{\Delta \xrightarrow{g} J \qquad \dfrac{J,\Phi \xrightarrow{h} C}{J,\Phi,A \longrightarrow C}\,\theta}{\Delta,\Phi,A \longrightarrow C}\,\sigma$$

(iii)

$$\dfrac{\dfrac{\Delta \xrightarrow{g} J \qquad J,\Phi,A,A \xrightarrow{h} C}{\Delta,\Phi,A,A \longrightarrow C}\,\sigma}{\Delta,\Phi,A \longrightarrow C}\,\kappa \qquad\qquad \dfrac{\Delta \xrightarrow{g} J \qquad \dfrac{J,\Phi,A,A \xrightarrow{h} C}{J,\Phi,A \longrightarrow C}\,\kappa}{\Delta,\Phi,A \longrightarrow C}\,\sigma$$

(iv)

$$\dfrac{\begin{array}{c} \dfrac{\Delta,A,A \xrightarrow{g} J \qquad J,\Phi \xrightarrow{h} C}{\Delta,A,A,\Phi \longrightarrow C}\,\sigma \\ \hline\hline \Delta,\Phi,A,A \longrightarrow C \end{array}}{\Delta,\Phi,A \longrightarrow C}\,\kappa \qquad\qquad \begin{array}{c} \dfrac{\dfrac{\Delta,A,A \xrightarrow{g} J}{\Delta,A \longrightarrow J}\,\kappa \qquad J,\Phi \xrightarrow{h} C}{\Delta,A,\Phi \longrightarrow C}\,\sigma \\ \hline\hline \Delta,\Phi,A \longrightarrow C \end{array}$$

(v)

$$\dfrac{\dfrac{\Delta \xrightarrow{g} J \qquad J,A,B,\Phi \xrightarrow{h} C}{\Delta,A,B,\Phi \longrightarrow C}\,\sigma}{\Delta,B,A,\Phi \longrightarrow C}\,\pi \qquad\qquad \dfrac{\Delta \xrightarrow{g} J \qquad \dfrac{J,A,B,\Phi \xrightarrow{h} C}{J,B,A,\Phi \longrightarrow C}\,\pi}{\Delta,B,A,\Phi \longrightarrow C}\,\sigma$$

(vi)

$$\dfrac{\dfrac{\Delta, A \xrightarrow{g} J \qquad J, B, \Phi \xrightarrow{h} C}{\Delta, A, B, \Phi \longrightarrow C}\;\sigma}{\Delta, B, A, \Phi \longrightarrow C}\;\pi
\qquad\qquad
\dfrac{\dfrac{\Delta, A \xrightarrow{g} J \qquad \dfrac{J, B, \Phi \xrightarrow{h} C}{B, J, \Phi \longrightarrow C}\;\pi}{B, \Delta, A, \Phi \longrightarrow C}\;\sigma}{\Delta, B, A, \Phi \longrightarrow C}$$

(vii)

$$\dfrac{\dfrac{\Delta, A, B \xrightarrow{g} J \qquad J, \Phi \xrightarrow{h} C}{\Delta, A, B, \Phi \longrightarrow C}\;\sigma}{\Delta, B, A, \Phi \longrightarrow C}\;\pi
\qquad\qquad
\dfrac{\dfrac{\Delta, A, B \xrightarrow{g} J}{\Delta, B, A \longrightarrow J}\;\pi \qquad J, \Phi \xrightarrow{h} C}{\Delta, B, A, \Phi \longrightarrow C}\;\sigma$$

(viii)

$$\dfrac{\dfrac{\Delta \xrightarrow{g} J_1 \qquad J_1, \Phi \xrightarrow{h} A}{\Delta, \Phi \longrightarrow A}\;\sigma \qquad \dfrac{\Delta \xrightarrow{g'} J_2 \qquad J_2, \Phi \xrightarrow{h'} B}{\Delta, \Phi \longrightarrow B}\;\sigma}{\Delta, \Phi \longrightarrow A \wedge B}\;{\rightarrow}{\wedge}$$

$$\dfrac{\dfrac{\Delta \xrightarrow{g} J_1 \qquad \Delta \xrightarrow{g'} J_2}{\Delta \longrightarrow J_1 \wedge J_2} \qquad \dfrac{\dfrac{J_1, \Phi \xrightarrow{h} A}{J_1 \wedge J_2, \Phi \longrightarrow A}\;{\wedge}{\rightarrow} \qquad \dfrac{J_2, \Phi \xrightarrow{h'} B}{J_1 \wedge J_2, \Phi \longrightarrow B}\;{\wedge}{\rightarrow}}{J_1 \wedge J_2, \Phi \longrightarrow A \wedge B}\;{\rightarrow}{\wedge}}{\Delta, \Phi \longrightarrow A \wedge B}\;\sigma$$

(ix)

$$\dfrac{\dfrac{\Delta \xrightarrow{g} J \qquad J, \Phi, A \xrightarrow{h} C}{\Delta, \Phi, A \longrightarrow C}\;\sigma}{\Delta, \Phi, A \wedge B \longrightarrow C}\;{\wedge}{\rightarrow}
\qquad\qquad
\dfrac{\Delta \xrightarrow{g} J \qquad \dfrac{J, \Phi, A \xrightarrow{h} C}{J, \Phi, A \wedge B \longrightarrow C}\;{\wedge}{\rightarrow}}{\Delta, \Phi, A \wedge B \longrightarrow C}\;\sigma$$

(x)

$$\dfrac{\dfrac{\dfrac{\Delta, A \xrightarrow{g} J \qquad J, \Phi \xrightarrow{h} C}{\Delta, A, \Phi \longrightarrow C}\;\sigma}{\Delta, \Phi, A \longrightarrow C}}{\Delta, \Phi, A \wedge B \longrightarrow C}\;{\wedge}{\rightarrow}
\qquad\qquad
\dfrac{\dfrac{\dfrac{\Delta, A \xrightarrow{g} J}{\Delta, A \wedge B \longrightarrow J}\;{\wedge}{\rightarrow} \qquad J, \Phi \xrightarrow{h} C}{\Delta, A \wedge B, \Phi \longrightarrow C}}{\Delta, \Phi, A \wedge B \longrightarrow C}$$

(xi)

$$\dfrac{\dfrac{\Delta \xrightarrow{g} J_1 \qquad J_1, \Phi, A \xrightarrow{h} C}{\Delta, \Phi, A \longrightarrow C}\,\sigma \qquad \dfrac{\Delta \xrightarrow{g'} J_2 \qquad J_2, \Phi, B \xrightarrow{h'} C}{\Delta, \Phi, B \longrightarrow C}\,\sigma}{\Delta, \Phi, A \vee B \longrightarrow C}\,\vee\!\rightarrow$$

$$\dfrac{\dfrac{\Delta \xrightarrow{g} J_1 \qquad \Delta \xrightarrow{g'} J_2}{\Delta \longrightarrow J_1 \wedge J_2}\,\rightarrow\!\wedge \qquad \dfrac{\dfrac{J_1, \Phi, A \xrightarrow{h} C}{J_1 \wedge J_2, \Phi, A \longrightarrow C}\,\wedge\!\rightarrow \qquad \dfrac{J_2, \Phi, B \xrightarrow{h'} C}{J_1 \wedge J_2, \Phi, B \longrightarrow C}\,\wedge\!\rightarrow}{J_1 \wedge J_2, \Phi, A \vee B \longrightarrow C}\,\vee\!\rightarrow}{\Delta, \Phi, A \vee B \longrightarrow C}\,\sigma$$

(xii)

$$\dfrac{\dfrac{\dfrac{\Delta, A \xrightarrow{g} J_1 \qquad J_1, \Phi \xrightarrow{h} C}{\Delta, A, \Phi \longrightarrow C}\,\sigma}{\Delta, \Phi, A \longrightarrow C} \qquad \dfrac{\dfrac{\Delta, B \xrightarrow{g'} J_2 \qquad J_2, \Phi \xrightarrow{h'} C}{\Delta, B, \Phi \longrightarrow C}\,\sigma}{\Delta, \Phi, B \longrightarrow C}}{\Delta, \Phi, A \vee B \longrightarrow C}\,\vee\!\rightarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Delta, A \xrightarrow{g} J_1}{\Delta, A \longrightarrow J_1 \vee J_2}\,\rightarrow\!\vee \qquad \dfrac{\Delta, B \xrightarrow{g'} J_2}{\Delta, B \longrightarrow J_1 \vee J_2}\,\rightarrow\!\vee}{\Delta, A \vee B \longrightarrow J_1 \vee J_2}\,\vee\!\rightarrow \qquad \dfrac{J_1, \Phi \xrightarrow{h} C \qquad J_2, \Phi \xrightarrow{h'} C}{J_1 \vee J_2, \Phi \longrightarrow C}\,\vee\!\rightarrow}{\Delta, A \vee B, \Phi \longrightarrow C}\,\sigma}{\Delta, \Phi, A \vee B \longrightarrow C}$$

(xiii)

$$\dfrac{\dfrac{\Delta \xrightarrow{g} J \qquad J, \Phi \xrightarrow{h} A}{\Delta, \Phi \longrightarrow A}\,\sigma}{\Delta, \Phi \longrightarrow A \vee B}\,\rightarrow\!\vee \qquad\qquad \dfrac{\Delta \xrightarrow{g} J \qquad \dfrac{J, \Phi \xrightarrow{h} A}{J, \Phi \longrightarrow A \vee B}\,\rightarrow\!\vee}{\Delta, \Phi \longrightarrow A \vee B}\,\sigma$$

(xiv)

$$\dfrac{\dfrac{\Delta \xrightarrow{g} J_1 \quad J_1,\Phi \xrightarrow{h} A}{\Delta,\Phi \longrightarrow A} \qquad \dfrac{\Delta \xrightarrow{g'} J_2 \quad J_2,\Phi,B \xrightarrow{h'} C}{\Delta,\Phi,B \longrightarrow C}}{\Delta,\Phi,A\supset B \longrightarrow C}$$

$$\dfrac{\dfrac{\Delta \xrightarrow{g} J_1 \quad \Delta \xrightarrow{g'} J_2}{\Delta \longrightarrow J_1\wedge J_2}{\to\wedge} \qquad \dfrac{\dfrac{J_1,\Phi \xrightarrow{h} A}{J_1\wedge J_2,\Phi \longrightarrow A}{\wedge\to} \quad \dfrac{J_2,\Phi,B \xrightarrow{h'} C}{J_1\wedge J_2,\Phi,B \longrightarrow C}{\wedge\to}}{J_1\wedge J_2,\Phi,A\supset B \longrightarrow C}{\supset\to}}{\Delta,\Phi,A\supset B \longrightarrow C}$$

(xv)

$$\dfrac{\dfrac{\dfrac{\Delta \xrightarrow{g} J_1 \quad J_1,\Phi \xrightarrow{h} A}{\Delta,\Phi \longrightarrow A}}{\Delta,\Phi \longrightarrow A} \qquad \dfrac{\dfrac{\Delta,B \xrightarrow{g'} J_2 \quad J_2,\Phi \xrightarrow{h'} C}{\Delta,B,\Phi \longrightarrow C}}{\Delta,\Phi,B \longrightarrow C}}{\Delta,\Phi,A\supset B \longrightarrow C}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{J_1,\Delta \xrightarrow{h} A \quad \dfrac{\Delta,B \xrightarrow{g'} J_2}{J_1,\Delta,B \longrightarrow J_2}}{J_1,\Delta,A\supset B \longrightarrow J_2}}{\Delta,A\supset B \longrightarrow J_1\supset J_2} \qquad \dfrac{\Phi \xrightarrow{g} J_1 \quad J_2,\Phi \xrightarrow{h'} C}{\Phi,J_1\supset J_2 \longrightarrow C}}{\Phi,\Delta,A\supset B \longrightarrow C}}{\Delta,\Phi,A\supset B \longrightarrow C}$$

(xvi)

$$\dfrac{\dfrac{\Delta \xrightarrow{g} J \quad J,\Phi,A \xrightarrow{h} B}{\Delta,\Phi,A \longrightarrow B}}{\Delta,\Phi \longrightarrow A\supset B}{\to\supset} \qquad \dfrac{\Delta \xrightarrow{g} J \quad \dfrac{J,\Phi,A \xrightarrow{h} B}{J,\Phi \longrightarrow A\supset B}{\to\supset}}{\Delta,\Phi \longrightarrow A\supset B}$$

The above pairs of derivations are in fact equivalent, although it is tedious to demonstrate this. Instead we refer the reader to Szabo (74) for the fact that if two derivations differ only by permutations of "mutually passive" inferences then the two derivations are equivalent, and we demonstrate the equivalence of pair (xvi) as an example.

We need to show that

$$\Omega(\hat{h}\circ(\hat{g}\wedge 1\wedge 1)\circ\nu) = \Omega(\hat{h}\circ\nu)\circ(\hat{g}\wedge 1).$$

By the naturality of ν the diagram

$$\begin{array}{ccc}
A\wedge(\bigwedge\Delta)\wedge(\bigwedge\Phi) & \xrightarrow{\ \nu\ } & (\bigwedge\Delta)\wedge(\bigwedge\Phi)\wedge A \\
\Big\downarrow{\scriptstyle 1\wedge g\wedge 1} & & \Big\downarrow{\scriptstyle g\wedge 1\wedge 1} \\
A\wedge J\wedge(\bigwedge\Phi) & \xrightarrow{\ \nu\ } & J\wedge(\bigwedge\Phi)\wedge A
\end{array}$$

commutes, and so

$$\Omega(\hat{h}\circ(\hat{g}\wedge 1\wedge 1)\circ\nu) = \Omega(\hat{h}\circ\nu\circ(1\wedge\hat{g}\wedge 1)).$$

But a short calculation shows that by the naturality of Ω

$$\Omega(\hat{h}\circ\nu\circ(1\wedge\hat{g}\wedge 1)) = \Omega(\hat{h}\circ\nu)\circ(\hat{g}\wedge 1).$$

4.10 Theorem

Craig's theorem is valid for $\underline{\mathcal{K}}_D$ in the sense of theorem 4.9.

Proof. The proof is identical to that given for theorem 4.9 except that for the induction step we must show, in addition, the equivalence of the following two derivations.

$$\dfrac{\dfrac{\Delta\xrightarrow{g}J \qquad J,\Phi,\neg A\xrightarrow{h}A}{\Delta,\Phi,\neg A\longrightarrow A}\,\sigma}{\Delta,\Phi\longrightarrow A}\,Nx_1 \qquad\qquad \dfrac{\Delta\xrightarrow{g}J \qquad \dfrac{J,\Phi,\neg A\xrightarrow{h}A}{J,\Phi\longrightarrow A}\,Nx_1}{\Delta,\Phi\longrightarrow A}\,\sigma$$

That is, we need to show

$$\circledcirc(\Omega(\hat{h}\circ(\hat{g}\wedge 1\wedge 1)\circ\nu)) = \circledcirc(\Omega(\hat{h}\circ\nu))\circ(\hat{g}\wedge 1).$$

But this follows immediately from the naturality of $\circledcirc$ and the demonstration of part (xvi) of the induction step of theorem 4.9.

4.11 Theorem

Craig's theorem is valid for $\underline{\mathcal{A}}_J$ in the sense of theorem 4.9.

Proof. The proof is identical to that given for theorem 4.9 except that for the induction step we must show, in addition, the equivalence of the following two derivations.

$$\dfrac{\dfrac{\Delta \xrightarrow{g} J \qquad J,\Phi \xrightarrow{h} \bot}{\Delta,\Phi \longrightarrow \bot}}{\Delta,\Phi \longrightarrow D}\,Fj \qquad\qquad \dfrac{\Delta \xrightarrow{g} J \qquad \dfrac{J,\Phi \xrightarrow{h} \bot}{J,\Phi \longrightarrow D}\,Fj}{\Delta,\Phi \longrightarrow D}$$

But this is trivial since by associativity of composition we have

$$\iota_D\circ(\hat{h}\circ(\hat{g}\wedge 1)) = (\iota_D\circ h)\circ(g\wedge 1).$$

4.12 Theorem

Craig's theorem is valid for $\underline{\mathcal{A}}_E$ in the sense of theorem 4.9.

Proof. The proof is completely analogous to that given for theorem 4.9.

Bibliography

The bibliographical conventions used here are adapted from the "Bibliography of the Theory of Models" of Addison, de Bouvere and Pitt (65). "Items in the bibliography are first arranged alphabetically by the last name of the author. The papers of a given author are then arranged chronologically by year of publication, and are cited in the articles by the last name of the author and the last two digits in the year of publication - e.g. 'Skolem [20]'. In those cases where a given author has more than one paper included for a given year, these items are arranged alphabetically by title and labeled, for example, 'Beth [53]', 'Beth [53a]', 'Beth [53b]', and 'Beth [53c]'. . . . Forthcoming items were labeled, for example, 'Keisler [a]', 'Keisler [b]', and 'Keisler [c]'." In this thesis we have preferred parentheses to square brackets for obvious reasons.

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